# Attack, defense, and the market for protection*

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**Abstract**

We model a market for protection. Demand is derived from a game with heterogeneous populations of attackers and defenders. Each attacker decides whether to carry out a risky attack. Defenders choose whether to pay for protection or remain exposed. We provide sufficient conditions for existence and uniqueness of a Nash equilibrium, characterized by the proportions of active attackers and protected defenders. An increased proportion of protected defenders reduces each defender's exposure, and hence, willingness to pay via two channels. Further, a decreased proportion of active attackers lowers the cost of protection to each defender. We analyse welfare and provide simulations.

**JEL classification:** C02; C62; C63; C70; D40; D60.

**Keywords:** attack; defense; population game; equilibrium; market for protection provision; welfare analysis; contagion; immunization; cybercrime and cybersecurity.

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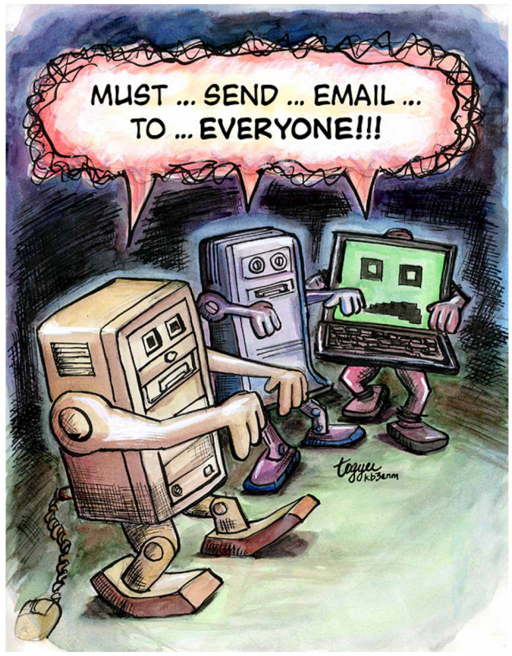


Figure 1: Zombie computers.

[Source: http://uvmzombies.blogspot.com.au/2013/02/computer-zombies.html]

# 1 Introduction

In many economic systems conflicts between two sets of agents may arise for the appropriation and the protection of valuable resources. When one set of agents unleashes attacks against another, the latter is left to bear the cost of being under attack. Initial attacks may provoke secondary attacks, which amplifies the outreach/consequences of the attacks, affecting the entire environment in which both attackers and defenders interact with one another. Examples of these conflicts can be found in cybercrime, the other side of which involves cybersecurity, or biological warfare demanding immunization, or the economics of tax evasion[1]

In these examples, prospective attackers assess their idiosyncratic gains from succeeding against the losses from failing. Similarly, defenders assess their idiosyncratic losses from attacks against the cost of investing in protection. Both, the defenders' individual and the aggregate investment in protection determine how **exposed** they are to such attacks. Conversely, the aggregate investment in protection together with the institutions in the economy determine how **adverse** the environment is for potential attackers. As the adversity increases, prospective attackers envisage an increased rate of failure from pursuing attacks,

[1]See Becker (1968), for a study of the economics of crime and punishment more in general.

and more attackers refrain from attacking. Similarly, an increase in the defenders' exposure to potential attacks increases their incentives to seek protection. Once defenders, in their attempt to avoid being 'sitting ducks', **demand** protection from specialized firms that **supply** it, a **market for protection** is formed.

These markets are special because they involve complex demand and supply side effects. Two channels form the demand side effect. First, there is a direct channel working as follows. When the proportion of protected defenders increases, all defenders' exposure reduces. Second, there is an indirect channel working as follows. When the proportion of protected defenders increases, the adversity attackers face increases, dissuading some of them from attacking, which also reduces the defenders' exposure. This reduction in exposure decreases the defenders' demand for protection.

There is a single channel supply side effect. As fewer attackers attack, the cost of individual protection decreases, increasing the supply of protection. We refer to the supply-side effect as 'economies of use'. These economies of use are different from economies of scale. Economies of use come about because the *use* of protection deters attackers and thereby leads to fewer modes of attack. By contrast, economies of scale come about solely from cost savings of producing a larger quantity, irrespective of how many units are consumed.

To study these markets, we construct a game with heterogeneous populations of attackers and defenders. Each attacker decides whether to perpetrate untargeted attacks. Attacking is risky; it may fail or succeed. Each defender chooses whether to pay for protection or remain exposed to an attack. If attacked, an unprotected defender risks suffering losses. We provide sufficient conditions for the existence and uniqueness of a Nash equilibrium, characterized by the proportion of attackers that attack and the proportion of defenders paying for protection. We explore the incentives for the private provision of protection to defenders, at a price, by a monopoly and by firms within Cournot and Bertrand oligopolies. We discuss the social welfare consequences of the private choices in the market for protection and provide simulations for such markets.

We find that, if protection is provided at a fixed price in a market, the is a unique Nash equilibrium in the population game. The demand for protection coming out of the population game is downward-sloping. Next we explicitly introduce the cost of providing protection and define welfare in this economy and show that any market solution is inefficient. We give sufficient conditions under which a monopolist charges a finite price. However, because for reasonable assumptions on the distributions of the defenders potential losses and the attackers potential gains demand can be convex, the monopoly problem need not be concave and therefore may have multiple local and even global maxima. Whether it does depends on the relative convexity of the demand and the cost of protection. We show that the monopolist may supply in the elastic, unit-elastic, or the inelastic portion of the demand, as a result of the relative size of the direct and indirect cost effect, that is, of the extent of the economies of use.

We characterize possible Cournot and Bertrand equilibria in the market for protection. For Cournot competition, we show that, contrary to standard Cournot models, the firms' supplied quantity of protection need not be strategic substitutes under all circumstances. They are strategic complements if the supply-side economies of use dominate the demand-side network effect. For Bertrand competition, we first show that multiple equilbria cannot be ruled out in general. We then provide a sufficient condition for the equilibrium to be unique. Roughly speaking, this condition holds if the demand-side network effect dominates the economies of use. We extend our analysis to the situation in which both the price and the quality of protection are the choice of a monopoly. All results obtained with perfect protection hold. In addition, we provide an optimality condition for quality provision of the monopolist. This condition resembles the Dorfman-Steiner condition for optimal advertising. Finally, we provide parametrized examples for our models for the market for protection, from monopoly to oligopoly, and simulate results within these environments for several distribution functions.

Many studies devote attention to aspects such as the reliability of the cyberspace and the feasibility of data encryption methods (e.g., in computer science). This may also involve focusing on the analysis of the underlying architecture of networks, connecting agents susceptible to attacks. The objective is to capture the agents' exposure to attacks, which is likely to depend on the degree of their connectedness within a network.[2] Other studies concentrate on the epidemiology aspect of contagion and its containment in the event of an outbreak/pandemic, therefore neglecting the perpetrated component of the attacks we are interested in capturing instead.[3] However, a comprehensive study of the economics of attacks and defense, looking at the incentives of attackers and defenders to respectively engage in attacks and in seeking protection at a price in a market environment is still missing. By allowing for alternative market structures for the provision of protection and alternative pricing schemes for the supply of protection, the present work attempts to overcome this gap.

[2]This is done, to study how connectedness affects the speed at which contagion can occur and spread, within those networks. Findings of these studies suggest ways to make a network more secure including how to design networks to better block attacks from propagating, as in Goyal and Vidier (2014). See also Chowdhury and Topolyan (2016) for a study of attack and defense in group contests.

[3]See Toxaerd (2009, 2015) for good examples of studies of the economics of non-perpetrated attacks and defense from an epidemiological standpoint.

## 2 The Model

Consider a simultaneous move, static game of complete information between two heterogeneous populations. There is a continuum, unit mass of defenders in population $J$ and a continuum, unit mass of attackers in population $I$. Each attacker $i \in I$, decides if she attacks or not whereas each defender $j \in J$ chooses to pay for protection against attacks or to save resources, risking to suffer the negative consequences of an attack. All players are utility maximizers. An attack is successful if and only if the defender did not pay for protection (an attack fails if and only if the defender paid for protection). Attackers cannot observe if a defender has protection or not. Let $\omega \in \Omega$ be a state of the world; that is, a collection of parameters, and $\Omega$ denote the set of all possible states; that is, the set of all feasible collections of parameters.

**Attackers** Let $F_X$ denote the continuous, atomless distribution of payoffs for a successful attack. As usual, $F_X(+\infty) = 1$. Attacker $i$ obtains a direct payoff of $x_i > 0$ if her attack succeeds. She also obtains an indirect payoff of $x_i$ from all unprotected defenders her target is directly or indirectly connected to during the attack.

If an attacker decides not to attack, her payoff is zero for sure. Denote the mass of defenders not taking protection be $\lambda$, with $0 \leq \lambda \leq 1$. For every $\lambda$, attacker $i$'s utility if she does not attack is given by:

$$U_i(\text{no attack} \mid \lambda) = 0.$$

An attacker loses payoff for each attack that fails. For each state $\omega \in \Omega$, $\beta(\lambda; \omega)$ is a function of the mass of defenders not taking protection, $\lambda$. For convenience, we will write just $\beta(\lambda)$, or simply $\beta$, if the context is clear. If player $i \in I$ decides to attack her utility is:

$$U_i(\text{attack} \mid \lambda) = x_i - \beta.$$

The *adversity function* $\beta(\lambda)$, capturing how difficult the attack environment common to all attackers is, enters the payoff component from attacking for all attackers, while the term $x_i$ depends on each attacker's idiosyncratic gain from succeeding in an attack. A natural assumption would be for $\beta$ to be positive, and weakly decreasing in $\lambda$.[4] An attacker $i \in I$ prefers to attack if $U_i(\text{attack} \mid \lambda) > U_i(\text{no attack} \mid \lambda)$; that is, if:

$$x_i > \beta(\lambda).$$

[4]Increasing or decreasing functions mean strictly increasing or decreasing. To refer to weakly monotonic function, the word "weak" is always written.

For each attacker, the aggregate behavior of other attackers is not important; she only cares about the aggregate behavior of defenders. Let $\chi$ be the mass of attackers choosing not to attack, with $0 \leq \chi \leq 1$. Then:

$$\chi = F_X(\beta(\lambda))\,.$$

**Defenders** For each state $\omega \in \Omega$ and mass of attackers choosing not to attack $\chi$, let $P(\chi, \lambda; \omega) > 0$ be the price of protection. If the context is clear, we write $P(\chi, \lambda)$ or just $P$. Defender $j$ obtains payoff $-P$ if she has paid for protection. It does not matter how large the population of attackers is or if protected defender $j$ is attacked many times, once, or not at all; if she paid for protection, her payoff is always $-P$. This formulation is compatible with full protection (for defenders that choose to protect themselves against a possible attack), costing each defender an amount $P$.[5] For every $\chi$:

$$V_j(\text{protection} \mid \chi) = -P.$$

Defender $j$ obtains payoff $-s_j < 0$ if she is attacked one or more times and does not have protection. Let $F_S$ denote the continuous distribution of costs for unprotected, attacked defenders. The payoff of the unprotected defender $j$ depends on being attacked (directly or indirectly). For now, we abstract from the process of direct and indirect consequences of attacks and consider the suffering from attacks to be captured by the *exposure function* $\delta = \delta(\chi, \lambda; \omega)$, decreasing with $\chi$, increasing with $\lambda$, and such that the payoff of being unprotected is:

$$V_j(\text{no protection} \mid \chi, \lambda) = -s_j\delta.$$

For every fixed $\omega$ and $\lambda$, as more attackers attack, $\chi$ decreases, function $\delta(\chi, \lambda)$ increases, and the payoff of being unprotected $V_j(\text{no protection} \mid \chi, \lambda)$ becomes more negative. For every fixed $\omega$ and $\chi$, as more defenders become unprotected, $\lambda$ increases, and then, $\delta$ increases, making the payoff of every unprotected defender to decrease. Defender $j$ prefers to pay for protection if $V_j(\text{protection} \mid \chi) > V_j(\text{no protection} \mid \chi, \lambda)$; that is, $-P > -s_j\delta$. Equivalently:

$$s_j > \frac{P(\chi, \lambda)}{\delta(\chi, \lambda)}.$$

Each defender $j$ cares about the aggregate behavior of both populations. Defenders $j$ such that $s_j \leq P(\chi, \lambda)/\delta(\chi, \lambda)$ prefer to take a chance and not pay for protection. The mass of unprotected defenders is:

$$\lambda = F_S\left(\frac{P(\chi, \lambda)}{\delta(\chi, \lambda)}\right).$$

[5] We examine the case of partial protection in an extension of the model, and otherwise maintain the full protection assumption for the rest of the analysis.

Consider the elasticities $\varepsilon_{P,\chi}$ and $\varepsilon_{\delta,\chi}$, respectively, of $P$ and $\delta$ with respect to $\chi$. Let $\varepsilon_{P,\lambda}$ and $\varepsilon_{\delta,\lambda}$ denote the elasticities of $P$ and $\delta$, respectively, with respect to $\lambda$. Formally:

$$\begin{aligned} \varepsilon_{P,\chi} &= \frac{\partial P}{\partial \chi}\frac{\chi}{P}, \qquad\qquad \varepsilon_{\delta,\chi} = \frac{\partial \delta}{\partial \chi}\frac{\chi}{\delta}, \\ \varepsilon_{P,\lambda} &= \frac{\partial P}{\partial \lambda}\frac{\lambda}{P}, \qquad\qquad \varepsilon_{\delta,\lambda} = \frac{\partial \delta}{\partial \lambda}\frac{\lambda}{\delta}. \end{aligned}$$

# 3 Attack-Defense Equilibrium

A function is smooth if it is differentiable infinitely many times. Consider the following collection of technical assumptions. Under these assumptions, there is a Nash equilibrium. These assumptions must hold for every state $\omega \in \Omega$.

**Assumption 1** *The distributions of $X$ and $S$ are continuous, atomless, have probability density functions (PDFs from now on) $f_X$ and $f_S$, respectively, and support contained on the set of non-negative real numbers. Moreover, $f_X(x_i) > 0$ and $f_S(s_j) > 0$, for every $x_i > 0$ and $s_j > 0$.*

**Assumption 2** *Function $(\chi, \lambda) \mapsto \delta(\chi, \lambda)$ is smooth and $0 < \delta(\chi, \lambda) < +\infty$, for every $0 < \chi < 1$ and $0 \leq \lambda \leq 1$. Moreover, $\partial\delta/\partial\chi < 0$ and $\partial\delta/\partial\lambda > 0$.*

This assumption implies that $\delta(\chi, \lambda)$ decreases with $\chi$, and increases with $\lambda$. Also, function $\delta(\chi, \lambda)$ is maximized at $(\chi, \lambda) = (0, 1)$.

**Assumption 3** *Function $(\chi, \lambda) \mapsto P(\chi, \lambda)$ is smooth and $P(\chi, \lambda) > 0$, for every $0 \leq \chi \leq 1$.*

**Assumption 4** *Function $\lambda \mapsto \beta(\lambda)$ is smooth; $\beta(\lambda) \geq 0$, for every $\lambda \geq 0$. For every $\lambda > 0$, $d/d\lambda\,[\beta] < 0$.*

This assumption implies that, for $\lambda > 0$, function $\lambda \mapsto \beta(\lambda)$ is strictly decreasing. Hence the function $\beta(\lambda)$ is minimized at $\lambda = 1$, and maximized in the limit, as $\lambda \to 0$.

**Remark 1** *Each defender has an idiosyncratic loss that, weighted by the exposure $\delta$, determines his willingness to pay. Similarly, each attacker has an idiosyncratic gain from attacking that, weighted with the fail-success-ratio $\beta$ to capture the adversity of the environment in which attacks are executed, determines his willingness to attack. In typical models, both are constants. Contrary to this, in our model, both the exposure and the adversity factors depend on the aggregate outcomes in the economy.*

An outcome for the game is a pair $(\chi, \lambda)$, where $\chi$ is the mass of attackers that do not attack and $\lambda$ is the mass of defenders that do not pay for protection. The outcome $(\chi^*, \lambda^*)$ is a Nash equilibrium if:

$$\chi^* = F_X(\beta(\lambda^*)), \tag{1}$$

$$\lambda^* = F_S\left(\frac{P(\chi^*, \lambda^*)}{\delta(\chi^*, \lambda^*)}\right). \tag{2}$$

The equilibrium $(\chi^*, \lambda^*)$ generates the correct incentives for the two populations. Deviations by subsets of players with zero measure are allowed because these deviations do not change incentives. Hence, uniqueness of equilibrium refers to uniqueness up to changes of behavior by zero measure subsets of players.

To economize on notation, let $\chi = F_X(\beta(\lambda))$, for every $0 \leq \lambda \leq 1$. Define the elasticity of $\chi$ with respect to $\lambda$ as

$$\varepsilon_{\chi,\lambda} = \frac{d\chi}{d\lambda}\frac{\lambda}{\chi} = f_X(\beta(\lambda))\frac{d\beta}{d\lambda}\frac{\lambda}{\chi}.$$

Consider the auxiliary variable $\overline{\varepsilon}$ defined by:

$$\overline{\varepsilon} = \frac{\lambda\delta(F_X(\beta(\lambda)), \lambda)}{P(F_X(\beta(\lambda)), \lambda) f_S\left(\frac{P(F_X(\beta(\lambda)),\lambda)}{\delta(F_X(\beta(\lambda)),\lambda)}\right)}.$$

Given the previous assumptions, it is clear that $\overline{\varepsilon} \geq 0$.

**Proposition 1** *Fix a state $\omega \in \Omega$. Suppose that assumptions 1 through 4 are satisfied. Then, the game with parameters characterized by state $\omega$ has at least one Nash equilibrium. In all Nash equilibria, $0 \leq \lambda^* \leq 1$ and $0 \leq \chi^* \leq 1$. There is a unique Nash equilibrium if for every $0 \leq \lambda \leq 1$ the following inequality is satisfied:*

$$\varepsilon_{P,\lambda} + \varepsilon_{P,\chi}\varepsilon_{\chi,\lambda} < \varepsilon_{\delta,\lambda} + \varepsilon_{\delta,\chi}\varepsilon_{\chi,\lambda} + \overline{\varepsilon}. \tag{3}$$

Because $\overline{\varepsilon} \geq 0$, regardless of the value of $0 \leq \lambda \leq 1$, then uniqueness of the Nash equilibrium holds if $\varepsilon_{P,\lambda} + \varepsilon_{P,\chi}\varepsilon_{\chi,\lambda} < \varepsilon_{\delta,\lambda} + \varepsilon_{\delta,\chi}\varepsilon_{\chi,\lambda}$, for every $0 \leq \lambda \leq 1$. The terms $\varepsilon_{P,\lambda}$ and $\varepsilon_{P,\chi}\varepsilon_{\chi,\lambda}$ capture the direct and the indirect (via changes in the mass of inactive attackers) sensitivity of the price to changes in the mass of unprotected defenders. Similarly, the terms $\varepsilon_{\delta,\lambda}$ and $\varepsilon_{\delta,\chi}\varepsilon_{\chi,\lambda}$ capture the direct and the indirect sensitivity of the exposure to changes in the mass of unprotected defenders. Therefore, uniqueness holds if the price of protection, $P$, is less sensitive to changes in the mass of unprotected defenders than the exposure, $\delta$, for every $0 \leq \lambda \leq 1$.

Regarding comparative statics, suppose there is a unique Nash equilibrium $(\chi^*, \lambda^*)$, with $0 < \lambda^* < 1$ and $0 < \chi^* < 1$. Then, the equilibrium number of unprotected defenders decreases (i.e., more defenders get protection) as the potential payoffs of successful attacks increases, or as the distribution of costs for unprotected, attacked defenders shifts right (increases). Intuitively, as potential attackers potential gains from successful attacks increase, the mass of active attackers will also increase. As a consequence, the risk of being unprotected is higher and more defenders will get protection.

Similarly, if the potential gains of successful attacks increase, fewer attackers stay inactive, and the risk of attacks increases. Hence, fewer defenders stay unprotected. The next result makes this intuition more precise by comparing distributions using a notion of strict first order stochastic dominance. The result has two parts. In the first, the distribution $S$ changes to $\hat{S}$ and the distribution $X$ stays the same. In the second part, the distribution $X$ changes to $\widehat{\widehat{X}}$ while the distribution $S$ stays the same.

**Corollary 1** *Suppose that assumptions 1 through 4 are satisfied. Suppose that condition (3) holds for every $0 \leq \lambda \leq 1$. Denote the unique Nash equilibrium $(\chi^*, \lambda^*)$ when the primitive distributions are $S$ and $X$. When only the distribution $S$ is perturbed, becoming $\widehat{S}$ (i.e., the distribution $X$ is unchanged), suppose that there is still a unique Nash equilibrium denoted $(\widehat{\chi}, \widehat{\lambda})$. When only the distribution $X$ is perturbed, becoming $\widehat{\widehat{X}}$ (i.e., the distribution $S$ is unchanged), suppose that there is still a unique Nash equilibrium denoted $(\widehat{\widehat{\chi}}, \widehat{\widehat{\lambda}})$.*

*(1) Suppose that the distribution $\widehat{S}$ dominates the distribution $S$ in the following precise sense: $F_{\widehat{S}}(s) < F_S(s)$, for all $s > 0$. Suppose that $X$ is unchanged. Then, the equilibrium mass of unprotected defenders decreases, $\widehat{\lambda} < \lambda^*$, and the mass of inactive attackers increases, $\widehat{\chi} > \chi^*$.*

*(2) Assume that function $P(\chi, \lambda)$ is weakly decreasing with $\chi$; that is, the price of protection weakly increases with the mass of active attackers. Suppose that the distribution $\widehat{\widehat{X}}$ dominates the distribution $X$ in the precise sense that $F_{\widehat{\widehat{X}}}(x) < F_X(x)$, for all $x > 0$. Suppose that distribution $S$ is unchanged. Then, the equilibrium mass of inactive attackers decreases, $\widehat{\widehat{\chi}} < \chi^*$, and the mass of unprotected defenders decreases, $\widehat{\widehat{\lambda}} < \lambda^*$.*

Section 9 contains some explicit examples of distributions. In these examples, the distributions are parametrized. Then, the domination relation described in Corollary 1 becomes simply the comparison of a parameter.

# 4 Cost of Protection Provision

Let $K$ be a positive integer number. Let there be a set $\mathcal{K} = \{1, \ldots, k, \ldots, K\}$ of providers of protection. These providers could be private firms, governments, or even clubs. We assume each defender is protected by only one provider. That is, if the mass of defenders protected by provider $k$ is given by $q_k$, then the total mass of protected defenders is given by $Q = \sum_{k \in \mathcal{K}} q_k$ and the mass of unprotected defenders is $\lambda = 1 - Q$.

The cost of a typical provider $k$, denoted $C_k(\chi, q_k)$, is a function of the mass of defenders protected by provider $k$, $q_k$, and the mass of attackers that choose not to attack, $\chi$.

**Assumption 5** *Each function $C_k(\chi, q_k)$ are smooth, with $C_k(\chi, q_k) > 0$, $\partial C_k / \partial \chi < 0$ and $\partial C_k / \partial q_k > 0$.*

This assumption implies that each function $C_k(\chi, q_k)$ decreases with $\chi$ and increases with $q_k$. In particular, it is maximized at $(\chi, q_k) = (0, 1)$. The aggregate cost to society when a mass $1 - \chi$ of attackers attack and firms $1, 2, \ldots, K$ provide $q_1, q_2, \ldots, q_K$ is the summation of the cost over all $k$, $\sum_{k \in \mathcal{K}} C_k(\chi, q_k)$.

Because attackers choose their actions optimally, the mass of inactive attackers is given by $\chi = F_X(\beta(\lambda))$. A protection provider's choice of protecting more defenders deters attackers, which lowers all providers' cost of protection. The term *economies of use* refers to this effect. This is the fundamental difference of the cost structure in the classical framework to ours: The cost function of every single provider depends on the aggregate outcomes in the population subgame between attackers and defenders.

**Development and Service Technology** To illustrate, assume each provider $k \in \mathcal{K}$ has a cost function with potentially two components: $C_D(\chi)$ and $q_k C_U(\chi)$. We shall refer to $C_D(\chi)$ as the *development cost* and to $C_U(\chi)$ as the *per-user cost* of protection or *service cost* to provide protection to each defender. Function $C_D(\chi)$ pertains to the development of protection measures and procedures that can be adopted by all clients. Take, for example, the development of software for protection. Function $C_U(\chi)$ is, for instance, the per-user cost of the building up and/or maintenance of defensive network infrastructures.

If firm $k$ incurs both service and development costs, its total cost of providing protection to a mass $q_k$ of defenders, given that a mass $1 - \chi$ of attackers are attacking, is:

$$C_k(\chi, q_k) = q_k C_U(\chi) + C_D(\chi). \tag{4}$$

The mass of protected defenders affects this total cost directly and indirectly. First, there is a *direct effect*: if the measure of protected defenders increase, the factor $q_k$ also increases; more users will be protected. Second, there is an *indirect effect*: as more defenders become protected, the payoffs of attacking diminish, potentially changing the incentives of a positive mass of attackers. Some attackers may switch their preferred action from attack to not attack. The mass of attackers not attacking, $\chi$, increases. In turn, this decreases the cost of protection *per (protected) defender/user*. To capture this indirect effect, we assume that

$C_U(\chi)$ and $C_D(\chi)$ are decreasing with $\chi$. Under fewer attacks, producing protection to each defender becomes cheaper; there need to be fewer specific modes of protection. Formally:

**Assumption 6** *The cost functions for the monopoly to protect each defender are decreasing with the quantity of attackers that prefer not to attack:*

$$\frac{d}{d\chi}[C_U] < 0, \qquad \frac{d}{d\chi}[C_D] < 0.$$

**Example 1** *Assume a unique provider, $k = 1$. This unique provider's cost is $C_1(\chi, q_1) = q_1 C_U(\chi) + C_D(\chi) = (1 - \lambda)C_U(\chi) + C_D(\chi)$.*

**Example 2** *Assume an industry in which only one provider, labelled $k = 1$, incurs development costs and all other providers use the developed technology and incur only a per-user cost. Then, $C_1(\chi, q_1) = C_D(\chi)$ and $C_k(\chi, q_k) = q_k C_U(\chi)$ for all other $k \neq 1$. Because attackers choose their actions optimally, the quantity choice $q_k$ of each protection provider $k \neq 1$ impacts the per-user and the development cost of* all *providers.*

*In this example, the aggregate cost of providing protection is given by*

$$C_D(\chi) + \sum_{k\in\mathcal{K}} q_k C_U(\chi) = C_D(\chi) + C_U(\chi) \sum_{k\in\mathcal{K}} q_k = C_D(\chi) + (1 - \lambda)C_U(\chi).$$

# 5 Welfare

This section analyses welfare in the face of attackers who react optimally, that is, given $\chi = F_X(\beta(\lambda))$. For that, define social welfare as the sum of the expected losses incurred by unprotected defenders and the cost of protecting the other defenders. Hence, for any given $\lambda$ and $\chi$, social welfare can be written as

$$W = -\int_{s_j=0}^{F_S^{-1}(\lambda)} s_j \delta(\chi, \lambda) f_S(s_j) ds_j - \sum_{k\in\mathcal{K}} C_k(\chi, q_k). \tag{5}$$

As always, the socially optimal allocation is characterized by two decisions: How to produce and how much to produce.

**Development and Service Technology** Assume the *Development and Service Technology* as described in section 4. Then, the development cost should be incurred only once. Let $q = 1 - \lambda$ denote the mass of protected defenders. The providers that incur costs $qC_U$ and $C_D$ need not to be the same. In this case, $\chi = F_X(\beta(\lambda))$, as attackers react optimally, and the welfare becomes simply:

$$W = -\int_{s_j=0}^{F_S^{-1}(\lambda)} s_j \delta(\chi, \lambda) f_S(s_j) ds_j - (1 - \lambda)C_U(\chi) - C_D(\chi). \tag{6}$$

With this formulation, welfare is a function of $\lambda$ only. Suppose that a social planner maximizes welfare by solving the problem:

$$\max_{0\le\lambda\le 1} \quad -\int_{s_j=0}^{F_S^{-1}(\lambda)} s_j\delta(\chi,\lambda)f_S(s_j)ds_j - (1-\lambda)C_U(\chi) - C_D(\chi).$$

The first order condition is:

$$\frac{dW}{d\lambda} = [C_U(\chi) - F^{-1}(\lambda)\delta(\chi,\lambda)] \overbrace{-\frac{d\delta}{d\lambda}\int_0^{F^{-1}(\lambda)} s_j f_S(s_j)ds_j}^{<0} \overbrace{-\left((1-\lambda)\frac{d}{d\lambda}[C_U] + \frac{d}{d\lambda}[C_D]\right)}^{<0}.$$

The derivatives $\frac{d}{d\lambda}[C_U]$ and $\frac{d}{d\lambda}[C_D]$ are positive because $dC_U/d\chi < 0$, $dC_D/d\chi < 0$, $f_X > 0$, $d\beta/d\lambda < 0$ and:

$$\begin{aligned}
\frac{d}{d\lambda}[C_U] &= \frac{dC_U}{d\chi}\frac{d\chi}{d\lambda} = \frac{dC_U}{d\chi} f_X(\beta(\lambda))\frac{d\beta}{d\lambda} > 0, \\
\frac{d}{d\lambda}[C_D] &= \frac{dC_D}{d\chi}\frac{d\chi}{d\lambda} = \frac{dC_D}{d\chi} f_X(\beta(\lambda))\frac{d\beta}{d\lambda} > 0.
\end{aligned}$$

The welfare is surely decreasing with $\lambda$ if $C_U(\chi) - F^{-1}(\lambda)\delta(\chi,\lambda) \le 0$.

# 6 Monopoly Provision of Protection

In this section we explore the incentives for a monopoly to provide protection to defenders for a payment of a uniform per-user price $P(\chi,\lambda) = p$. The timing is as follows. The monopoly publicly announces a price. After observing the price, all defenders and attackers simultaneously choose their actions in the population subgame described in section 4. As usual, the solution is by backwards induction and, hence, the monopoly anticipates the outcome of the population subgame for each possible price and announces the profit-maximizing price. Attackers and defenders react optimally to this price.

## 6.1 Revenue

In order to find the monopoly solution, let us first write expressions for the change in the equilibrium values $\chi^*$ and $\lambda^*$ with changes in the price charged by the monopolist. For each price $p > 0$, the equilibrium level $\lambda^*$ is characterized by equation (8). By plugging $\lambda^*$ into the right-hand side of equation (7), the equilibrium $\chi^*$ is calculated:

$$\chi^* = F_X(\beta(\lambda^*)), \tag{7}$$

$$\lambda^* = F_S\left(\frac{p}{\delta\left(F_X(\beta(\lambda^*)), \lambda^*\right)}\right). \tag{8}$$

Given Proposition 1, for each price $p > 0$, this equilibrium $(\chi^*, \lambda^*)$ exists, is unique and interior. Attackers and defenders take the monopolistic price as given when they make their decisions. Hence, $\varepsilon_{P,\chi} = \varepsilon_{P,\lambda} = 0$. This implies that the sufficient condition for uniqueness obtained in equation (3) simplifies to $0 < \varepsilon_{\delta,\lambda} + \varepsilon_{\delta,\chi}\varepsilon_{\chi,\lambda} + \overline{\varepsilon}$. This inequality always holds as $\varepsilon_{\delta,\lambda} > 0$, $\varepsilon_{\delta,\chi} < 0$, $\varepsilon_{\chi,\lambda} < 0$, and $\overline{\varepsilon} > 0$.

The next results explain how the equilibrium variables $\chi^*$ and $\lambda^*$ change with marginal increases in the price $p$.

**Lemma 1** *The functions $\chi^*(p)$ and $\lambda^*(p)$ are differentiable and their derivatives are given by*

$$\frac{d}{dp}[\lambda^*] = \frac{f_S\left(\frac{p}{\delta(\chi^*,\lambda^*)}\right)\delta(\chi^*,\lambda^*)}{\delta^2(\chi^*,\lambda^*) + p f_S\left(\frac{p}{\delta(\chi^*,\lambda^*)}\right)\left(\frac{\partial\delta}{\partial\chi} f_X(\beta(\lambda^*))\frac{d}{d\lambda}[\beta] + \frac{\partial\delta}{\partial\lambda}\right)}, \tag{9}$$

*and*

$$\frac{d}{dp}[\chi^*] = \frac{f_X(\beta(\lambda^*))\frac{d}{d\lambda}[\beta]\, f_S\left(\frac{p}{\delta(\chi^*,\lambda^*)}\right)\delta(\chi^*,\lambda^*)}{\delta^2(\chi^*,\lambda^*) + p f_S\left(\frac{p}{\delta(\chi^*,\lambda^*)}\right)\left(\frac{\partial\delta}{\partial\chi} f_X(\beta(\lambda^*))\frac{d}{d\lambda}[\beta] + \frac{\partial\delta}{\partial\lambda}\right)}. \tag{10}$$

The next result proves that the demand for protection decreases as the price goes up.

**Lemma 2** *The equilibrium share of the defenders not buying protection is always increasing and the equilibrium share of the attackers not attacking is always decreasing with the price:*

$$\frac{d}{dp}[\lambda^*] > 0,$$
$$\frac{d}{dp}[\chi^*] < 0.$$

## 6.2 Costs

The total cost of providing protection depends on the equilibrium values of the share of defenders buying protection $(1-\lambda^*)$ and the share of attackers attacking $(1-\chi^*)$. Assume the monopolist's cost takes the form of Example 1 in subsection 4. Every $p > 0$ induces a corresponding equilibrium of the population subgame $(\chi^*, \lambda^*)$. Let $q^* = 1-\lambda^*$. By an abuse of notation, when referring to the cost of the monopoly, drop subscript $k$ and write:

$$C(\chi^*, q) = q^* C_U(\chi^*) + C_D(\chi^*) = (1-\lambda^*)C_U(\chi^*) + C_D(\chi^*).$$

Let $MC$ and $AC$ denote the marginal and average cost and assume that attackers react optimally to any mass of protected defenders. Then:

$$\begin{aligned}
MC &= -\frac{d}{d(1-\lambda^*)}[(1-\lambda^*)C_U(F_X(\beta(\lambda^*))) + C_D(F_X(\beta(\lambda^*)))] \\
&= C_U(F_X(\beta(\lambda^*))) - \left((1-\lambda^*)\frac{dC_U}{d\chi} + \frac{dC_D}{d\chi}\right)\frac{d}{d\lambda}[F_X(\beta(\lambda^*))]. \\
AC &= C_U(F_X(\beta(\lambda^*))) + \frac{C_D(F_X(\beta(\lambda^*)))}{1-\lambda^*}.
\end{aligned}$$

When the monopolist increases the price $p$, it will also increase the costs of protection provision *per defender*. Indeed, because $d\chi^*/dp < 0$, $\partial C_U/\partial \chi < 0$ and $\partial C_D/\partial \chi < 0$, then:

$$\frac{d}{dp}[C_U^*] = \frac{\partial C_U}{\partial \chi}\frac{d}{dp}[\chi^*] > 0, \tag{11}$$

$$\frac{d}{dp}[C_D^*] = \frac{\partial C_D}{\partial \chi}\frac{d}{dp}[\chi^*] > 0. \tag{12}$$

Consider the impact of a price increase on this cost. On the one hand, lemma 2 states that an increase in the price will increase the equilibrium share of unprotected defenders, which means they need not be protected, lowering the cost. This is the usual direct effect of price on the demand. On the other hand, the increase in the equilibrium share of unprotected defenders decreases the share of inactive attackers, which increases the cost of protecting each of the defenders. This is an indirect effect, a novelty of this model. The two effects push the total cost in opposite directions. Formally, the total cost's derivative with respect to the price is:

$$\frac{d}{dp}\left[(1-\lambda^*)C_U(\chi^*) + C_D(\chi^*)\right] = \underbrace{\left((1-\lambda^*)\frac{\partial C_U}{\partial \chi} + \frac{\partial C_D}{\partial \chi}\right)\frac{d}{dp}[\chi^*]}_{\text{indirect effect, >0}} \underbrace{-C_U(\chi^*)\frac{d}{dp}[\lambda^*]}_{\text{direct effect, <0}}.$$

In order to find out which effect dominates, one needs to compare:

$$\left((1-\lambda^*)\frac{\partial C_U}{\partial \chi} + \frac{\partial C_D}{\partial \chi}\right)\frac{d}{d\lambda}[\chi^*]\frac{d}{dp}[\lambda^*] \quad \text{and} \quad C_U(\chi^*)\frac{d}{dp}[\lambda^*].$$

Because $d\lambda^*/dp > 0$, using equation (49), we can conclude that the direct effect is the dominating one if and only if:

$$\left((1-\lambda^*)\frac{\partial C_U}{\partial \chi} + \frac{\partial C_D}{\partial \chi}\right)\frac{d}{d\lambda}[\chi^*] - C_U(\chi^*) < 0. \tag{13}$$

Inspection of the formula for the marginal cost yields the following result.

**Proposition 2** *The marginal cost is positive (negative) if and only if the direct effect dominates (is dominated by) the indirect effect.*

**Lemma 3** *Suppose that* $C_D \geq 0$. *Assume that* $\frac{d}{d\chi}[C_U] < 0$ *or* $\frac{d}{d\chi}[C_D] < 0$. *Then, the average cost is strictly larger than the service cost, and both of them are strictly larger than the marginal cost:*

$$AC \geq C_U > MC, \tag{14}$$

*with* $AC = C_U$ *if* $C_D = 0$.

Define the price elasticity of protection demand as

$$\varepsilon_{Q^*,p} = \frac{dQ^*}{dp}\frac{p}{Q^*}. \tag{15}$$

**Lemma 4** *Suppose that* $C_D > 0$, $\frac{d}{d\chi}[C_U] < 0$ *or* $\frac{d}{d\chi}[C_D] < 0$. *Then, the average cost is strictly increasing in* $p$. *More precisely:*

$$\frac{dAC}{dp} = -\frac{\varepsilon_{Q^*,p}}{p}(AC - MC) \tag{16}$$

$$\frac{dAC}{dp} > 0. \tag{17}$$

This lemma implies that the average cost is decreasing in the quantity of protected defenders. The next corollary formalizes it.

**Corollary 2** *Suppose that* $C_D > 0$, $\frac{d}{d\chi}[C_U] < 0$ *or* $\frac{d}{d\chi}[C_D] < 0$. *Then, the average cost is strictly decreasing in* $1 - \lambda^*$. *More precisely:*

$$\frac{dAC}{d(1-\lambda)} < 0.$$

If $C_D > 0$, then the market is a natural monopoly because decreasing average costs are present for any level of protection provided by the monopoly.

Assume, to the contrary, that $C_D = 0$ and $dC_U/d\chi < 0$. Then, decreasing average costs over the whole range of the demand would be driven by decreasing marginal costs, whether one or many providers were to provide protection, violating the condition for a natural monopoly to exist. In this second scenario decreasing average costs are simply a consequence of the increased aggregate use of protection.

## 6.3 Profit Maximization

When the monopolist sets a price $p$ for protection, the total revenue is given by $(1-\lambda^*)p$. Then its profit is given by $\pi = (1-\lambda^*)(p - C_U(\chi^*)) - C_D(\chi^*)$ and its profit maximization problem is:

$$\max_{p>0} \ \{(1-\lambda^*)(p - C_U(\chi^*)) - C_D(\chi^*)\},$$

where $\chi^*$ and $\lambda^*$ are determined by:

$$\chi^* = F_X(\beta(\lambda^*)) \, ,$$
$$\lambda^* = F_S\left(\frac{p}{\delta(\chi^*, \lambda^*)}\right) .$$

Call the lowest price that covers the service costs $\underline{p}$.

**Assumption 7** *Function $C_U(\chi)$ is such that there exists a unique $\underline{p}$ such that $C_U(\chi^*(\underline{p})) = \underline{p}$, $C_U(\chi^*(p)) > p$ for all $p < \underline{p}$ and $C_U(\chi^*(p)) < p$ for all $p > \underline{p}$.*

**Assumption 8** *Both partial derivatives of the exposure are bounded. Formally, $\delta_\lambda < +\infty$ and $\delta_\chi > -\infty$.*

**Assumption 9** $\lim_{p\to+\infty}(p - C_U(\chi^*(p)))\frac{f_S(p/\delta)}{1-F_S(p/\delta)} > 1$.

**Assumption 10** *The expected damage of successful attack is finite, $\int_0^{+\infty} sf(s)ds < +\infty$, and there is a number $\overline{s} > 0$ such that $f_S$ is weakly decreasing for every $s \geq \overline{s}$.*

**Proposition 3** *Under Assumptions 8 – 10, the monopoly maximizes its profit by choosing a finite price $p \geq \underline{p}$.*

**Remark 2** *The profit-maximizing price need not be unique because, under very reasonable assumptions, $\frac{d^2}{dp^2}[\lambda^*] < 0$; that is, the demand for protection, $1 - \lambda^*$, is downward sloping, but convex.*

The derivative of the monopolist's profit with respect to price is:

$$\frac{d\pi}{dp} = (1-\lambda^*) - \left(p\frac{d}{dp}[\lambda^*] + \left((1-\lambda^*)\frac{\partial C_U}{\partial \chi} + \frac{\partial C_D}{\partial \chi}\right)\frac{d}{dp}[\chi^*] - C_U(\chi^*)\frac{d}{dp}[\lambda^*]\right), \tag{18}$$

where the derivative $\partial C/\partial\chi$ is calculated at $\chi^* = F_X(\beta(\lambda^*))$. By equation (10), the first order condition $d\pi/dp = 0$ becomes:

$$\frac{1-\lambda^*}{\frac{d}{dp}[\lambda^*]} = p + \left((1-\lambda^*)\frac{\partial C_U}{\partial \chi} + \frac{\partial C_D}{\partial \chi}\right)\frac{d\chi^*}{d\lambda} - C_U(\chi^*), \tag{19}$$

where $d\chi^*/d\lambda$ is calculated at $\lambda = \lambda^*$, and $\partial C/\partial\chi$ is calculated at $\chi^* = F_X(\beta(\lambda^*))$. The expression in parenthesis on the right-hand side is the sum of the price $p$ and the left-hand side of inequality (13), which is the difference of the indirect and the direct effects. This difference can be negative (so the direct effect is dominating), but when we add $p$, the result must be positive because $1 - \lambda^* > 0$ and $d\lambda^*/dp > 0$. Hence, any price that satisfies the first order condition sets an upper bound on how much the direct effect can dominate the indirect effect, both effects evaluated at the equilibrium of the population subgame for this price.

The sign of the elasticity $\varepsilon_{Q^*,p}$ is the opposite of the sign of $d\lambda^*/dp$. Some algebra leads to:

$$\frac{1-\lambda^*}{p\frac{d}{dp}[\lambda^*]} = \frac{q^*}{p\frac{d}{dp}[1-q^*]} = \frac{-1}{\frac{p}{q^*}\frac{d}{dp}[q^*]} = \frac{-1}{\varepsilon_{Q^*,p}}. \tag{20}$$

It is possible to express the monopoly mark-up in relation to the price elasticity of the demand for protection. Dividing both sides of equation (19) by $p$ and using equation (20) imply that the profit maximization condition $d\pi/dp = 0$ can be rewritten to resemble the standard Lerner rule:

$$\frac{-1}{\varepsilon_{Q^*,p}} = \frac{p - \overbrace{\left(C_U(\chi^*) - \left((1-\lambda^*)\frac{\partial C_U}{\partial \chi} + \frac{\partial C_D}{\partial \chi}\right)\frac{d\chi^*}{d\lambda}\right)}^{\text{marginal cost}}}{p}. \tag{21}$$

Remember, the marginal defender's willingness to pay depends of the distribution of losses and the defenders' exposure, $p = F_S^{-1}(\lambda)\delta(F_X(\beta(\lambda)), \lambda)$. Hence, it is possible to decompose the price elasticity of demand into two parts, the exposure elasticity of demand, $\varepsilon_{Q,\delta} = \frac{dQ}{d\delta}\frac{\delta}{Q}$ and the loss distribution elasticity of demand, $\varepsilon_{Q,F_S^{-1}} = \frac{dQ}{dF_S^{-1}}\frac{F_S^{-1}}{Q}$. Decomposing the price elasticity of demand leads to:

$$\frac{1}{\varepsilon_{Q,p}} = \frac{1}{\varepsilon_{Q,\delta}} + \frac{1}{\varepsilon_{Q,F_S^{-1}}}. \tag{22}$$

Therefore, the modified Lerner rule can be expressed as:

$$\frac{-1}{\varepsilon_{Q,\delta}} + \frac{-1}{\varepsilon_{Q,F_S^{-1}}} = \frac{p - \left(C_U(\chi^*) - \left((1-\lambda^*)\frac{\partial C_U}{\partial \chi} + \frac{\partial C_D}{\partial \chi}\right)\frac{d\chi^*}{d\lambda}\right)}{p}. \tag{23}$$

The elasticity of the demand with respect to the loss distribution on the left-hand side of equation (23) derives from the classical effect: Individuals have their idiosyncratic loss and as the quantity is expanded, price needs to be lowered. The exposure elasticity of demand derives from the (demand side) network effect. The right-hand side of equation (23) represents the monopoly mark-up over the service cost plus a positive term related to economies of use. The economies of use effect is weakly positive, $\left((1-\lambda^*)\frac{\partial C_U}{\partial \chi} + \frac{\partial C_D}{\partial \chi}\right)\frac{d\chi^*}{d\lambda} \geq 0$ and, hence, attenuates the mark-up.

Repeating these steps with inequalities instead of equalities leads to:

$$\frac{d\pi}{dp} \geq 0 \Leftrightarrow -\varepsilon_{Q^*,p}\frac{p - MC}{p} \leq 1. \tag{24}$$

At every price $p > 0$ where the monopoly profit is zero, $\pi = 0$, the rate of change of the profit with respect to the price is positive (zero, negative) if and only if the rate of change of the average cost with respect to the price is smaller than (equal to, greater than) one.

**Remark 3** *In all monopolistic markets, we have:*

$$\frac{d\pi}{dp} - \frac{\pi}{q^*}\frac{dq^*}{dp} = q^*\left(1 - \frac{dAC}{dp}\right). \tag{25}$$

*In particular, if* $\frac{dAC}{dp} > 1$*, then* $\frac{d\pi}{dp} < 0$*. If* $\frac{d\pi}{dp} > 0$*, then* $\frac{dAC}{dp} < 1$*. Now, suppose* $\pi = 0$*. By equations (24) and (25),*

$$\frac{d\pi}{dp} \geq 0 \Leftrightarrow -\varepsilon_{Q^*,p}\frac{p - MC}{p} \leq 1 \Leftrightarrow \frac{dAC}{dp} \leq 1,$$

*with equality holding on the left-hand side if and only if it holds on the right-hand side.*

*Next, let* $p^M$ *be any price that satisfies the first order condition of the monopoly profit maximization problem. Then:*

$$\pi > 0 \Leftrightarrow \frac{dAC(p^M)}{dp} < 1.$$

*Finally, let* $dAC/dp = 1$*. Then:*

$$\frac{d\pi}{dp}\frac{p}{\pi} = \varepsilon_{Q^*,p}.$$

*If the profit is positive and* $dAC/dp = 1$*, then the monopolist should increase its price. If it is negative and* $dAC/dp = 1$*, then the monopolist should decrease its price.*

**Remark 4** *In our model,* $dAC/dp > 0$*. Hence,* $q^*(1 - dAC/dp) < q^*$*. By equation (25), this puts an upper bound on the marginal profit with respect to price:*

$$\frac{d\pi}{dp} < \frac{\pi}{q^*}\frac{dq}{dp} + q^*.$$

**Proposition 4** *Let* $p$ *be any price that satisfies the first order condition of the monopoly profit maximization problem. Then, the following three statements are equivalent:*

1. *At* $p$*, the marginal cost of protection is strictly positive (strictly negative; zero);*

2. *At* $p$*, the direct cost dominate the economies of use (the economies of use dominate the direct cost; the direct cost equal the economies of use);*

3. *At* $p$*, the demand for protection is elastic (inelastic; unit-elastic).*

**Example 3** *Suppose that the service cost is identically zero, $C_U(\chi) = 0$, for every $\chi \in [0, 1]$. Then, the cost to protect consists of only a fixed cost for the development of specific modes of protection. Clearly, this is a natural monopoly.*

*Interestingly, even though the cost are exclusively fixed cost in the sense that, once the monopolist has found a protection for a particular attacker's mode of attack, it can use it for every defender without additional cost, the marginal cost is not zero. It is negative.*

*As before, assume this development cost $C_D(\chi)$ is decreasing in $\chi$. Equation (12) reveals that the development cost increases with the price. In this example, this means that the total cost increases with the price. If the monopolist increases its "output", i.e., lowers its price and provides protection to more defenders, then the number of attackers that does not attack in equilibrium increases, lowering the monopolist's cost.*[6]

*Suppose the monopoly lowers the price. Then, the number of defenders served increases. Hence, the total cost of providing defense* falls. *The* marginal cost *of production is* negative. *The Lerner rule in equation (21) becomes simply:*

$$\frac{-1}{\varepsilon_{Q^*,p}} = \frac{p + \frac{\partial C_D}{\partial \chi}\frac{d\chi^*}{d\lambda}}{p}.$$

*Contrary to the standard monopoly solution, the monopoly mark-up is greater than unity and the monopolist will produce in the inelastic part of the demand, $|\varepsilon_{Q^*,p}| < 1$.*

**Example 4** *Assume there are no development costs, i.e., $C_D(\chi) = 0$ for every $\chi \in [0, 1]$. Then the Lerner rule in equation (21) becomes simply:*

$$\frac{-1}{\varepsilon_{Q^*,p}} = \frac{p - C_U(\chi^*) + (1-\lambda^*)\frac{\partial C_U}{\partial \chi}\frac{d\chi^*}{d\lambda}}{p}.$$

*The term $-C_U(\chi^*) + (1-\lambda^*)\frac{\partial C_U}{\partial \chi}\frac{d\chi^*}{d\lambda} > 0$ if and only if $C_U(\chi^*) < -q^*\frac{dC_U}{dq}$, or, put differently, if and only if $-(dC_U/dq)/(C_U/q^*) > 1$. In this case, once more, contrary to the standard monopoly solution, the monopoly mark-up is greater than unity and the monopolist will produce in the inelastic part of the demand, $|\varepsilon_{Q^*,p}| < 1$.*

## 6.4 Induced Welfare under Monopoly

Let $W_J$ denote the welfare of the defenders, for any set price $p$:

$$W_J = -\int_{s_j=0}^{F_S^{-1}(\lambda^*)} s_j \delta(\chi^*, \lambda^*) f_S(s_j) ds_j - (1-\lambda^*)p. \tag{26}$$

[6] With a negative marginal cost of production and free disposal of any quantity produced, one would optimally produce an infinite quantity and dispose of everything that has not been consumed. Note that, in our model, free disposal does not hold because the cost-relevant quantity is not determined by production decisions but by equilibrium consumer behavior.

**Proposition 5** *Suppose Assumptions 1-4 hold. Then, for every $p > 0$, the defenders' welfare is strictly decreasing in the price. Formally, $dW_J/dp < 0$.*

Define the social welfare under monopoly as the sum of the defenders' welfare and the monopolist's profit, $W = W_J + \pi$. Since the revenues of providing protection and the defenders' expenses of purchasing protection cancel out when summing them up, social welfare can simply be written as

$$W = -\int_{s_j=0}^{F_S^{-1}(\lambda^*)} s_j \delta(p) f_S(s_j) ds_j - (1-\lambda^*) C_U(\chi^*) - C_D(\chi^*). \tag{27}$$

**Proposition 6** *Under Assumptions 8 – 10, the price that maximizes welfare is strictly smaller than any price $p^M$ that maximizes the monopoly profit.*

Stronger than that, we can show that, for all prices for which the monopolist recovers the service cost, welfare is decreasing in price.

**Proposition 7** *Suppose Assumption 2, in particular $\frac{\partial}{\partial \lambda}[\delta(\chi, \lambda)] > 0$, holds or Assumption 2, in particular $\frac{\partial}{\partial \chi}[\delta(\chi, \lambda)] < 0$, and Assumption 4, in particular $\frac{d}{d\lambda}[\beta(\lambda)] < 0$, hold or Assumption 6 holds. Then for every $p \geq \underline{p}$, welfare is strictly decreasing in the price.*

**Proposition 8** *Suppose Assumption 8 holds. Suppose that when all defenders are protected and no attackers attack, the direct cost effect outweighs the indirect cost effect; that is, the marginal cost is positive at $(\chi, \lambda) = (1, 0)$. Then, the welfare maximizing price of protection is strictly greater than zero. Mathematically, the welfare maximizing price is strictly greater than zero if:*

$$C_U(1) > \left( \frac{\partial C_U(1)}{\partial \chi} + \frac{\partial C_D(1)}{\partial \chi} \right) \frac{d}{d\lambda}[\chi^*(0)].$$

**Remark 5 (Policy: Subsidizing the development cost in a competitive industry)** *Suppose the government covers the development costs and gives the developed product for free to a free-entry industry that provides the protection service. If this avoids duplication of development costs, it may be welfare-enhancing. However, Proposition 7 implies that the resulting equilibrium is still inefficient. Welfare can be improved by supplying more protection at a lower price. This holds even if the service and development cost functions are not decreasing in $\chi$. It also holds if the exposure, $\delta$, was only a function of the number of active attackers and not – as we assumed – of the protected defenders, i.e., if there were no indirect attacks.*

**Remark 6 (Policy: Subsidizing the development cost in a monopoly)** *Suppose the government covers the development costs and gives the developed product for free to a monopoly.*

*Then, as long as the development cost depends on the measure of active attackers, welfare is decreased compared to the situation in which the monopolist covers the development cost. In the monopolist's first order condition the derivative of the development cost enters negatively, implying this result.*

**Remark 7 (Policy: Subsidizing the service cost in a monopoly)** *Suppose the government grants a subsidy of $z$ per unit of service in a monopoly. Then, the monopolist's first order condition changes to:*

$$\frac{d\pi}{dp} = (1-\lambda^*) - \left( p\frac{d}{dp}[\lambda^*] + \left( (1-\lambda^*)\frac{\partial C_U}{\partial \chi} + \frac{\partial C_D}{\partial \chi} \right) \frac{d}{dp}[\chi^*] - (C_U(\chi^*) - z)\, \frac{d}{dp}[\lambda^*] \right).$$

# 7 Oligopolistic Provision of Protection

## 7.1 Cournot Competition

Assume an oligopolistic market with $K$ identical profit-maximizing firms that compete by choosing quantities $q_k \geq 0$, for $k \in \{1, \cdots, K\}$. Assume consumers perceive their protection services as homogeneous.

Suppose that when a mass of $1-\chi$ attackers are active, firm $k$'s cost function is $C_k(\chi, q_k) = q_k C_U(\chi)$. The aggregate quantity is $Q = \sum_{k=1}^{K} q_k$. Then, $\lambda = 1-Q$ is the mass of unprotected defenders, and $\chi = F_X\left(\beta(1-Q)\right)$ is the mass of inactive attackers. The market price $p$ is determined in equilibrium by the following equation:

$$1 - Q = F_S\left( \frac{p}{\delta\left(F_X\left(\beta(1-Q)\right), 1-Q\right)} \right).$$

Then, the price is $p = F_S^{-1}(1-Q)\delta\left(F_X\left(\beta(1-Q)\right), 1-Q\right)$. The profit of each firm $k$ is $\pi_k = q_k p - C_k(\chi, q_k) = q_k(p - C_U(\chi))$. Hence:

$$\pi_k = q_k F_S^{-1}(1-Q)\delta\left(F_X\left(\beta(1-Q)\right), 1-Q\right) - q_k C_U\left(F_X\left(\beta(1-Q)\right)\right). \tag{28}$$

The marginal profit of firm $k$ is:

$$\frac{d\pi_k}{dq_k} = F_S^{-1}\delta + q_k\delta\frac{d}{dq_k}[F_S^{-1}] + q_k F_S^{-1}\frac{d}{dq_k}[\delta] - q_k\frac{d}{dq_k}[C_U] - C_U.$$

Thus, firm $k$'s first order condition $d\pi_k/dq_k = 0$ is:

$$\underbrace{F_S^{-1}\delta + q_k\delta\frac{d}{dq_k}[F_S^{-1}] - C_U}_{\text{traditional effects}} + \underbrace{q_k\left(F_S^{-1}\frac{d}{dq_k}[\delta] - \frac{d}{dq_k}[C_U]\right)}_{\text{novel effects}} = 0.$$

Consider the network effect, $q_k F_S^{-1} d\delta/dq_k$. This effect is negative, it attenuates each firm's output reaction to its competitors' quantities. Now, consider the economies of use, $dC_U/dq_k$. This effect is positive, it amplifies each firm's output reaction to its competitors' quantities.

Figure 2 illustrates the two effects' impact on a firm's best response to an output reduction of their competitors in the Cournot model.

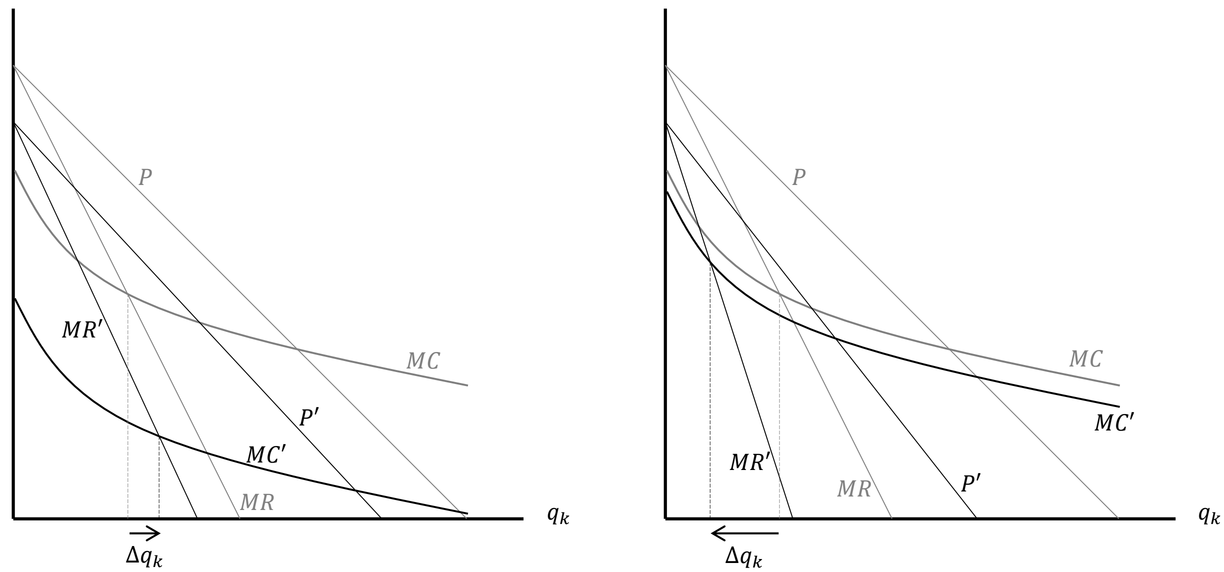


Figure 2: Response to an output increase of $\Delta Q$. Right: When the network effect dominates the economies of use, quantities are strategic substitutes. Left: When economies of use dominate the network effect, quantities are strategic complements. $P$, $MR$ and $MC$ are firm $k$'s inverse demand, marginal revenue and marginal cost before the rival's quantity increase by $\Delta Q$. $P'$, $MR'$ and $MC'$ are firm $k$'s inverse demand, marginal revenue and marginal cost after the rival's quantity increase by $\Delta Q$.

Simplification of the firms' first order conditions leads to the following equilibrium condition, which resembles the Lerner rule:

$$\frac{1}{K}\left(\frac{-1}{\varepsilon_{Q,p}}\right) = \frac{p - \left(C_U - q_k \frac{d}{dq_k}[C_U]\right)}{p}. \tag{29}$$

Using the elasticity decomposition in equation (22), it is easy to visualize the two novel effects in the model.

**Proposition 9** *Suppose all firms in a Cournot oligopoly have the same cost function,* $C_k(\chi, q_k) = q_k C_U(\chi)$*, where* $Q = K q_k$ *and* $\chi = F_X(\beta(1-Q))$*. Then, at the Cournot equilibrium the* *following condition holds:*

$$\frac{1}{K}\left(\frac{-1}{\varepsilon_{Q,\delta}}+\frac{-1}{\varepsilon_{Q,F_S^{-1}}}\right)=\frac{p-\left(C_U-q_k\frac{d}{dq_k}[C_U]\right)}{p}.$$

The exposure elasticity, $\varepsilon_{Q,\delta}$, captures the network effects in the demand. Everything else equal, the stronger the network effects, the higher the markup over the traditional marginal cost. The economies of use, $q_k\frac{d}{dq_k}[C_U]$, has the opposite impact. The stronger the economies of use, the lower is the markup over the traditional marginal cost.

**Remark 8 (Alternative cost structures: development costs)** *We could consider alternative cost structures. If, for example, the Cournot competitors were to incur a development cost as considered in equation (4) in section 4, then their first order condition, $d\pi_k/dq_k=0$, would change to:*

$$F_S^{-1}\delta+q_k\delta\frac{d}{dq_k}[F_S^{-1}]+q_kF_S^{-1}\frac{d}{dq_k}[\delta]-q_k\frac{d}{dq_k}[C_U]-C_U-\frac{d}{dq_k}[C_D]=0.$$

*Assuming once more $dC_D/d\chi<0$, the additional term, $-\frac{d}{dq_k}[C_D]$, is negative. Hence, holding everything else fixed, this modification lowers individual firm output and equilibrium output and increases equilibrium prices, all assuming $C_D$ is not prohibitively high so that firms produce positive output.*

**Remark 9 (Alternative cost structures: service costs)** *So far, we have assumed that, for a given share of inactive attackers, a firm's total service cost is proportional to the share of the mass of defenders it protects. Suppose, instead, that the cost function of each firm $k$ is:*

$$C_k(q_k,\chi)=g(q_k)C_U(\chi),$$

*for some differentiable function $g(q_k)$ with $dg/dq_k>0$. So far, $g(q_k)=q_k$. With the more general formulation, the Cournot competitors' first order condition $d\pi_k/dq_k=0$ would change to:*

$$F_S^{-1}\delta+q_k\delta\frac{d}{dq_k}[F_S^{-1}]+q_kF_S^{-1}\frac{d}{dq_k}[\delta]-g(q_k)\frac{d}{dq_k}[C_U]-C_U\frac{d}{dq_k}[g]=0.$$

*Assume, for example, $g(q_k)=\sqrt{q_k}$. Then $g(q_k)\geq q_k$ and $dg/dq_k\geq 0$. Alternatively, one could assume $g(q_k)=q_k^2$. Then $g(q_k)\leq q_k$ and $dg/dq_k\geq 0$.*

## 7.2 Bertrand Competition

Consider an oligopolistic market with $K$ identical firms that choose prices $p_k\geq 0$, for $k\in\{1,\ldots,K\}$. Suppose that consumers perceive their protection services as homogeneous. Further, assume that each firm's cost function only consists of service costs.

**Assumption 11** *$C_k(\chi, q_k) = q_k C_U(\chi)$, $C_U(0) < +\infty$, and $C_U(1) > 0$.*

The profit of each firm $k$ is $\pi_k = q_k p_k - q_k C_U(\chi)$, where:

$$q_k = \begin{cases} 0, & \text{if firm } k \text{ does not have the lowest price;} \\ \frac{1-\lambda^*(p)}{m}, & \text{if } k \text{ is one of } m \text{ lowest price firms.} \end{cases}$$

Further assume the following restrictions on $\beta(\lambda)$ and $\delta(\chi, \lambda)$.

**Assumption 12** *$\lim_{\lambda \to 0} \beta(\lambda) = +\infty$, $\beta(1) = 0$, $\delta(0,1) < +\infty$, and $\delta(1,0) > 0$.*

**Proposition 10** *Suppose Assumptions 11 and 12 hold.*

1. *Then there exists an equilibrium price $p$ such that $p = C_U(\chi^*(p))$.*

2. *Denote $\underline{p} = \min\{p | \, p = C_U(\chi^*(p))\}$. Then the equilibrium is unique if and only if $\frac{d}{dp}\left[C_U\left(\chi^*\left(\underline{p}\right)\right)\right] < 1$, irrespective of how many members $\{p | \, p = C_U(\chi^*(p))\}$ has.*

Figure 3 illustrates the existence and uniqueness of the Bertrand equilibrium.

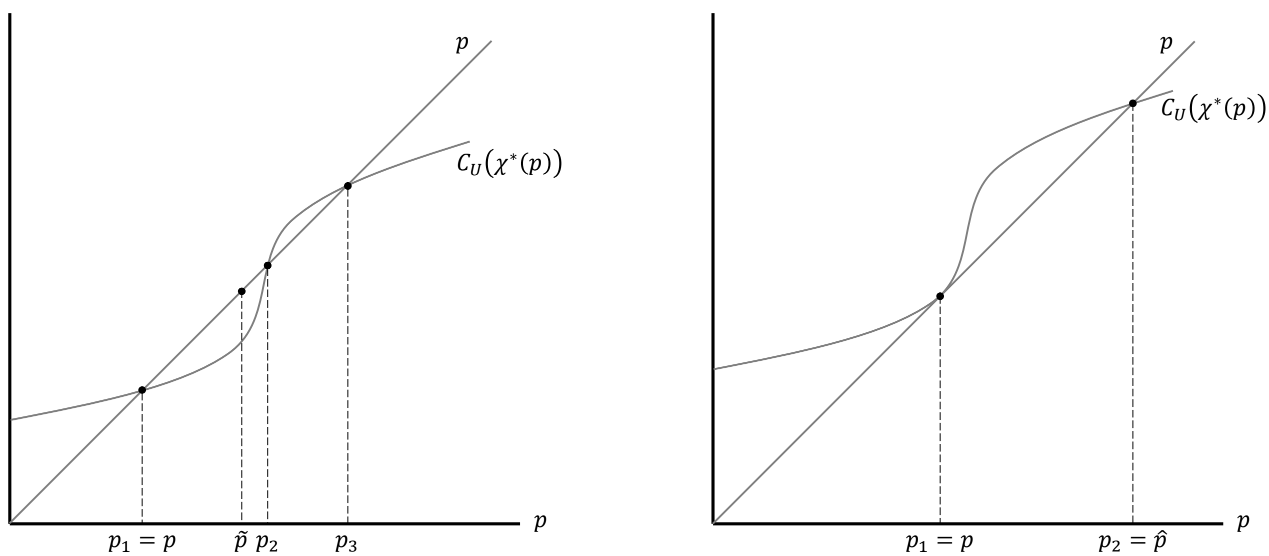


Figure 3: Existence and uniqueness of the Bertrand equilibrium. Left: Equilibrium is unique at $p_1 = \underline{p}$. $p_2$ and $p_3$ are not equilibria: firms would have an incentive to undercut to a price $\widetilde{p} < p_2$ as this gives strict positive profits. Right: Both $p_1 = \underline{p}$ and $p_2 = \widehat{p}$ are equilibria.

Define the elasticity of the service cost with respect to the mass of inactive attackers, denoted $\varepsilon_{C_U,\chi}$, and the elasticity of the price with respect to the mass of inactive attackers, denoted $\varepsilon_{p,\chi}$, as follows:

$$\varepsilon_{C_U,\chi} = \frac{d}{d\chi}\left[C_U\left(\chi^*\left(\underline{p}\right)\right)\right] \frac{\chi^*(p)}{C_U(\chi^*(p))},$$

$$\varepsilon_{p,\chi} = \frac{1}{\frac{d}{dp}\left[\chi^*\left(\underline{p}\right)\right]} \frac{\chi^*(p)}{p}.$$

**Corollary 3** *A sufficient condition for the equilibrium to be unique is* $C_U(\chi)\varepsilon_{C_U,\chi} > p\varepsilon_{p,\chi}$ *for all* $\chi$.

There are multiple equilibria if $\frac{d}{d\chi}\left[C_U\left(\chi^*\left(\underline{p}\right)\right)\right]\frac{d}{dp}\left[\chi^*\left(\underline{p}\right)\right] = 1$ and $\{p|\, p = C_U(\chi^*(p))\}$ has several members.

**Corollary 4** *In any equilibrium,* $C_U\varepsilon_{C_U,\chi} \geq p\varepsilon_{p,\chi}$.

Proposition 7 implies that any Bertrand equilibrium in the present model is inefficient. Indeed, our average cost is decreasing in the mass of protected defenders and, therefore, the marginal cost is smaller than the average cost everywhere. This implies in particular that, in any Bertrand equilibrium, the price is strictly larger than the marginal cost of protection.

**Corollary 5** *Suppose the assumptions from Proposition 7 hold. Then any Bertrand equilibrium in the Attack-and-Defense model is inefficient.*

This has important policy implications. Suppose a social planner is able to raise non-distrotive lump-sum taxes. Assume further, this social planner's development cost is not higher than that in the private sector. Suppose such a social planner supplies the Bertrand oligopolists with the developed protection technology such that there is no duplication of the development costs. Even in this case, Bertrand competition leads to an inefficient outcome. A subsidy would solve the problem. Referring to equation 6 in section 5, we find the optimal subsidy, denoted $\zeta$, is given by:

$$\zeta = C_U(\beta(\lambda^o)) - F_S^{-1}(\lambda^o)\delta(F_X(\beta(\lambda^o)), \lambda^o),$$

where

$$\lambda^o = \arg\max_{\lambda}\left\{-\int_0^{F_S^{-1}(\lambda)} s\delta(F_X(\beta(\lambda)), \lambda)f_s(s_j)ds_j - (1-\lambda)C_U(F_X(\beta(\lambda))) - C_D(F_X(\beta(\lambda)))\right\}.$$

Because the average cost is decreasing everywhere, the optimal subsidy is strictly larger than the difference between marginal cost and the price in any, including the lowest-price, laissez-faire Bertrand equilibrium.

There could be alternative, neither better nor worse, ways of implementing the efficient outcome. We refrain from characterising them. Furthermore, of course there are important political economy aspects of the subsidy solution. These aspects are well beyond the scope of the present paper and are left for further research.

**Remark 10 (Alternative Cost Structures)** *When there are development costs or when the service cost is not linear in $q_k$, the existence of a Nash equilibrium becomes a more delicate issue. In fact, the existence of a Nash equilibrium in the classical Bertrand competition model is an active area of research. The results on this direction depend on assumptions on fixed costs and on the sub-additivity of the cost function. For more details, see Saporitiy and Coloma (2010) and the references therein.*

# 8 Partial Protection

## 8.1 Partial Protection Setup

Consider a modified monopoly model that allows for partial protection. Suppose that the monopolist can choose the level of protection $0 \leq \phi \leq 1$ that protected defenders obtain. Full protection is obtained when $\phi = 1$. In general, protection is effective with probability $\phi$ only.

Assume that the utility of a defender $j$ who buys protection is given by:

$$V_j(\text{Protection}) = -p - (1-\phi)\delta(\chi, \lambda; \phi)s_j. \tag{30}$$

As before, assume the utility of defender $j$ who does not buy protection is:

$$V_j(\text{No Protection}) = -\delta(\chi, \lambda; \phi)s_j. \tag{31}$$

Suppose that $\beta(\lambda; \phi) \geq 0$, $\frac{\partial \beta}{\partial \lambda} < 0$ and $\frac{\partial \beta}{\partial \phi} > 0$. Let the utility of an active attacker be given by:

$$U_i(\text{Attack}) = x_i - \beta(\lambda; \phi) \tag{32}$$

The population game equilibrium is given by:

$$\begin{aligned} \chi^* &= F_X\left(\beta(\lambda^*; \phi)\right), \quad \text{and} \\ \lambda^* &= F_S\left(\frac{p}{\phi\delta(\chi^*, \lambda^*; \phi)}\right). \end{aligned}$$

Under assumptions 1–3 together with the assumptions on $\beta(\lambda; \phi)$ made here, this equilibrium can easily be shown to exist and to be unique. The equilibrium of the population game implies

$$p = \phi\delta(F_X(\beta(\lambda^*; \phi)), \lambda^*; \phi)F_S^{-1}(\lambda^*).$$

Next, suppose that the cost of protection depends both on the share of unprotected defenders, $\lambda$, the share of inactive attackers, $\chi$, and the level of protection, $\phi$. Assume a modified development and service technology leading to the following cost function

$$C(\chi, \lambda, \phi) = (1-\lambda)C_U(\chi, \phi), \tag{33}$$

with $\partial C_U/\partial\chi < 0$ and $\partial C_U/\partial\phi > 0$.

Then, the monopolist's profit maximization problem is given by:

$$\max_{\lambda,\phi}\left\{(1-\lambda)\left(\phi\delta F_S^{-1}(\lambda) - C_U(\chi,\phi)\right)\right\}. \tag{34}$$

## 8.2 Solving the Model of Monopoly with Partial Protection

The monopolist's first order conditions are given by:

$$\frac{\partial\pi}{\partial\lambda} = (1-\lambda)\phi\frac{d}{d\lambda}\left[\delta F_S^{-1}\right] - \phi\delta F_S^{-1} - (1-\lambda)\frac{\partial C_U}{\partial\chi}\frac{d\chi}{d\lambda} + C_U = 0, \tag{35}$$

$$\frac{\partial\pi}{\partial\phi} = \delta F_S^{-1} + \phi\frac{d}{d\phi}[\delta F_S^{-1}] - \frac{\partial C_U}{\partial\chi}f_X\frac{\partial\beta}{\partial\phi} - \frac{\partial C_U}{\partial\phi} = 0. \tag{36}$$

Equation (35) can be rewritten as:

$$\frac{1}{\phi}\left(C_U - (1-\lambda)\frac{\partial C_U}{\partial\chi}\frac{d\chi}{d\lambda}\right) = \delta F_S^{-1} - (1-\lambda)\frac{d}{d\lambda}[\delta F_S^{-1}]. \tag{37}$$

The first order condition with respect to $\lambda$ leads to the same Lerner rule we obtained earlier:

$$-\frac{1}{\varepsilon_{Q,p}} = \frac{p - \left(C_U - (1-\lambda)\frac{\partial C_U}{\partial\chi}\frac{d\chi}{d\lambda}\right)}{p}.$$

Equation (36) implies likely imperfect protection if the cost impact of improving protection efficacy is large. It implies likely perfect protection if the payoff from attacking reacts strongly with the efficacy of protection.

Define the elasticity of the demand with respect to the level of protection, denoted $\varepsilon_{Q,\phi}$, by:

$$\varepsilon_{Q,\phi} = \frac{dQ}{d\phi}\frac{\phi}{Q}. \tag{38}$$

Then, equation (36) can be rewritten as:

$$\delta F_S^{-1} + \phi\frac{d}{d\phi}[\delta F_S^{-1}] = \frac{\partial C_U}{\partial\chi}f_X\frac{\partial\beta}{\partial\phi} + \frac{\partial C_U}{\partial\phi}.$$

Some algebra leads to:

$$\frac{dp}{d\phi}\frac{\phi}{p} = \frac{\phi\frac{dC_U}{d\phi}}{p}.$$

Multiplying top and bottom of the leaf-hand side by $Q$ and using the chain rule $dp/d\phi = (dp/dQ)(dQ/d\phi)$ leads to:

$$\begin{aligned}\frac{dp}{dQ}\frac{Q}{p}\frac{dQ}{d\phi}\frac{\phi}{Q} &= \frac{\phi\frac{dC_U}{d\phi}}{p}\\ \frac{\varepsilon_{Q,\phi}}{\varepsilon_{Q,p}} &= \frac{\phi\frac{dC_U}{d\phi}}{p}.\end{aligned}$$

The optimal efficacy of protection is characterized by an equality of the ratio of the efficacy elasticity of demand over the price elasticity of demand and the ratio of the "efficacy expenditure" for a given user over the revenue from each user. This condition resembles Dorfman and Steiner's (1954) condition for optimal advertising expenditures.

How does the optimal $\lambda$ depend on the dependence of the willingness to pay and the cost on the efficacy of protection? Take the derivative of equation (35) with respect to $\phi$. This is positive if

$$\overbrace{\phi^2\left((1-\lambda)\frac{d^2}{d\lambda d\phi}[\delta F_S^{-1}]-\frac{d}{d\phi}[\delta F_S^{-1}]\right)}^{\text{willingness to pay impact of efficacy increase}}-\left(\overbrace{(1-\lambda)\frac{\partial C_U}{\partial\chi}\frac{d\chi}{d\lambda}}^{\text{economies of use}}+C_U\right)>$$
$$\underbrace{\phi\left((1-\lambda)\frac{\partial^2 C_U}{\partial\chi\partial\phi}\frac{d\chi}{d\lambda}-\frac{\partial}{\partial\phi}C_U\right)}_{\text{cost impact of efficacy increase}}$$

Take the realistic case of a positive cross-partial derivative of $C_U$. Then the monopolist's marginal profit with respect to the quantity increases in the efficacy of protection, $\phi$, as long as the economies of use are not larger than the cost impact of improving protection efficacy. In this case, the monopolist's quantity choice depends positively on $\phi$. If the economies of use are larger than the cost impact of improving the efficacy of protection, its quantity choice depends negatively on $\phi$.

# 9 Networked Defenders under Attack

This section specializes on the main model of attack and defense by incorporating elements of a network of defenders that may be attacked both directly and indirectly. Indirect attacks to a defender may come from one of her connections that was successfully attacked.

## 9.1 Setup

Assume that each target is randomly connected to a share $\kappa$ of other defenders and that this is independent of the protection decisions, with $0 \leq \kappa \leq 1$.

### 9.1.1 Attackers

The attacker obtains payoff normalized to $-1$ for each direct attack that fails, and $-(1-\xi)$, for each indirect attack that fails, where $0 \leq \xi \leq 1$ is a measure of the relative safety of indirect attacks compared to the safety of direct attacks. An attacker may also suffer from failed indirect attacks, but maybe not as much as she suffers from a failure in a direct attack. If $\xi = 1$, indirect attacks do not cause any extra loss to the attacker; that is, indirect attacks are totally safe. The only risk for an attacker comes from defense in the primary computers that are attacked. Indirect attacks hurt only the secondary computer, not the original attacker.

The other extreme case is $\xi = 0$. Each failure in a secondary attack causes as much loss to the attacker as a failure in a direct attack. Indirect attacks are as unsafe as direct ones.

If an attacker decides not to attack, her payoff is zero for sure. Denote the mass of defenders not taking protection be $\lambda$, with $0 \leq \lambda \leq 1$. Then for every $\lambda$ an attacker's utility if she does not attack is given by:

$$U_i(\text{no attack} \mid \lambda) = 0.$$

A primary attack triggers a stochastic number of secondary/indirect attacks. If player $i \in I$ decides to attack her utility is:

$$U_i(\text{attack} \mid \lambda) = -(1-\lambda) + \lambda Q_i, \tag{39}$$

where:

$$Q_i = x_i + \kappa(\lambda Q_i - (1-\lambda)(1-\xi)),$$

Variable $Q_i$ represents the payoff of attacker $i$ after a successful first attack, net of fixed cost. He gets $x_i$ and goes on to make another attack, an indirect one. This new attack finds a target with probability $\kappa$. With probability $1-\kappa$, no new target is found and attacker $i$ gets only $Q_i = x_i$. The second attack leads to the payoff $\lambda Q_i - (1-\lambda)(1-\xi)$: with probability $\lambda$ it is a success, leading to a return to $Q_i$; and, with probability $1-\lambda$, this secondary attack fails and the attacker loses $(1-\xi)$. Isolating $Q_i$ leads to:

$$Q_i = \frac{x_i - \kappa(1-\lambda)(1-\xi)}{1-\lambda\kappa}.$$

Plugging this back into equation (39), the payoff of attacking becomes:

$$\begin{aligned} U_i(\text{attack} \mid \lambda) &= -(1-\lambda) + \lambda \frac{x_i - \kappa(1-\lambda)(1-\xi)}{1-\lambda\kappa} \\ &= \frac{\lambda x_i - (1-\lambda)(1-\kappa\lambda\xi)}{1-\lambda\kappa}. \end{aligned}$$

If indirect attacks are completely safe, $\xi = 1$, then:

$$U_i(\text{attack} \mid \lambda) = \frac{\lambda x_i - (1-\lambda)(1-\lambda\kappa)}{1-\lambda\kappa}. \tag{40}$$

If indirect attacks are as unsafe as direct ones, $\xi = 0$, then the payoff of attacking becomes:

$$U_i(\text{attack} \mid \lambda) = \frac{\lambda x_i - (1-\lambda)}{1-\lambda\kappa}.$$

#### 9.1.2 Incentives for Attackers

An attacker $i \in I$ prefers to attack if $U_i(\text{attack} \mid \lambda) > U_i(\text{no attack} \mid \lambda)$, that is, if:

$$\frac{\lambda x_i - (1-\lambda)(1-\lambda\kappa\xi)}{1-\lambda\kappa} > 0.$$

Solving this inequality for $x_i$ results in:

$$x_i > (1-\lambda)\left(\frac{1}{\lambda} - \kappa\xi\right) := \beta(\lambda). \tag{41}$$

Let $\chi$ be the mass of attackers choosing not to attack, with $0 \leq \chi \leq 1$. This mass of attackers that prefer not to attack is given by:

$$\chi = F_X\left((1-\lambda)\left(\frac{1}{\lambda} - \kappa\xi\right)\right).$$

If $\kappa = 0$, there are no indirect attacks because defenders are not connected. In this case, $Q_i = x_i$ and inequality (41) becomes simply $x_i > (1-\lambda)/\lambda$.

#### 9.1.3 Defenders

The mass of attackers that play attack is $1-\chi$. Fix an exogenous price of protection $p > 0$. Defender $j$ obtains payoff $-p$ if she has paid for protection. It does not matter how large the population of attackers is or if protected defender $j$ is attacked many times, once, or not at all; if she paid for protection, her payoff is always $-p$. For every $\chi$:

$$V_j(\text{protection} \mid \chi) = -p.$$

Defender $j$ obtains payoff $-s_j < 0$ if she is attacked one or more times and does not have protection. Let $F_S$ denote the continuous distribution of costs for unprotected, attacked defenders. Assume that the probability of being attacked directly is $1-\chi$. The payoff of the unprotected defender $j$ depends on being attacked (directly or indirectly). So:

$$V_j(\text{no protection} \mid \chi) = (1-\chi)(-s_j) + \chi Y_j, \tag{42}$$

where:

$$Y_j = \kappa\lambda(1-\chi)(-s_j) + \kappa\lambda\chi Y_j.$$

Solving this equation for $Y_j$ results in:

$$Y_j = \frac{\kappa\lambda(1-\chi)(-s_j)}{1-\kappa\lambda\chi}.$$

Variable $Y_j$ represents the payoff of an unprotected defender $j$ if she is not directly attacked. If $\kappa\lambda > 0$, she may be indirectly attacked. Hence, $Y_j = 0$ if $\kappa\lambda = 0$, but $Y_j < 0$ if $\kappa\lambda > 0$.

Plugging this back into equation (42) and simplifying leads to:

$$V_j(\text{no protection} \mid \chi) = \frac{(1-\chi)(-s_j)}{1-\kappa\lambda\chi}. \tag{43}$$

### 9.1.4 Incentives for Defenders

Defender $j$ prefers to pay for protection if $V_j(\text{protection} \mid \chi) > V_j(\text{no protection} \mid \chi)$, that is, $-p > -s_j\delta$, where:

$$\delta = \frac{1-\chi}{1-\kappa\lambda\chi}.$$

Equivalently, $s_j > p/\delta$, or:

$$s_j > \frac{p(1-\kappa\lambda\chi)}{1-\chi} = \frac{p}{\delta(\chi,\lambda)}. \tag{44}$$

Defenders $j$ such that $s_j \leq p/\delta$ prefer to take a chance and do not pay for protection. The mass of unprotected defenders is $\lambda = F_S(p/\delta)$:

$$\lambda = F_S\left(\frac{p(1-\kappa\lambda\chi)}{1-\chi}\right).$$

For $\kappa = 0$, this model approaches the one without externalities from getting protection. In this case, condition (44) becomes $s_j > p/(1-\chi)$.

## 9.2 Existence and Uniqueness of Equilibrium

To show existence and uniqueness of an equilibrium in the population subgame, notice that in the specialized model Assumptions 2 and 4 hold. Both functions $\beta(\lambda)$ and $\delta(\chi,\lambda)$ are smooth. Also, $\beta(\lambda) \geq 0$, $0 < \delta(\chi,\lambda) < +\infty$, $d\beta/d\lambda < 0$, $\partial\delta/\partial\delta\chi < 0$, $\partial\delta/\partial\lambda > 0$, and:

$$\beta(\lambda) = (1-\lambda)\left(\tfrac{1}{\lambda} - \kappa\xi\right) \geq 0, \quad \tfrac{d}{d\lambda}[\beta] = -\tfrac{1-\lambda\kappa\xi}{\lambda^2};$$
$$\delta(\chi,\lambda) = \tfrac{1-\chi}{1-\kappa\lambda\chi}, \quad \tfrac{\partial\delta}{\partial\chi} = -\tfrac{1+\kappa\lambda}{(1-\kappa\lambda\chi)^2}; \quad \tfrac{\partial\delta}{\partial\lambda} = \tfrac{(1-\chi)\kappa\chi}{(1-\kappa\lambda\chi)^2}.$$

Furthermore, for a monopolist setting a positive uniform price also assumption 3 holds. Hence, the following corollary follows from Proposition 1.

**Corollary 6** *Under assumption 1, there exists a unique Nash equilibrium in the population subgame. This equilibrium is such that, for any finite price $p$, $0 < \lambda^* < 1$ and $0 \leq \chi^* \leq 1$.*

*The equilibrium values are given by*

$$\lambda^* = F_S\left(\frac{p(1-\kappa\lambda^*\chi^*)}{1-\chi^*}\right),$$
$$\chi^* = F_X\left((1-\lambda^*)\left(\frac{1}{\lambda^*}-\kappa\xi\right)\right).$$

## 9.3 Comparative Statics

The next result explains how the equilibrium changes with the parameters of the model.

**Proposition 11** *Suppose that $F_S(P) < 1$ and $0 < F_X(0) < 1$. Then:*

1. *As the price of protection, $p$, increases, the equilibrium share of attackers that do not attack, $\chi^*$, decreases and the equilibrium share of defenders that do not pay for protection, $\lambda^*$, increases:*

$$\frac{d\chi^*}{dp} \leq 0,$$
$$\frac{d\lambda^*}{dp} \geq 0.$$

2. *As the probability of contagion, $\kappa$, increases, the equilibrium share of attackers that do not attack, $\chi^*$, decreases and the equilibrium share of defenders that do not pay for protection, $\lambda^*$, decreases:*

$$\frac{d\chi^*}{d\kappa} \leq 0,$$
$$\frac{d\lambda^*}{d\kappa} \leq 0.$$

3. *As the safety of indirect attacks, $\xi$ increases, the equilibrium share of attackers that do not attack, $\chi^*$, increases and the equilibrium share of defenders that do not pay for protection, $\lambda^*$, decreases:*

$$\frac{d\chi^*}{d\xi} \geq 0,$$
$$\frac{d\lambda^*}{d\xi} \leq 0.$$

Higher safety of indirect attacks induces (weakly) more attacks and (weakly) less protection. More expensive protection leads to (weakly) more attacks and (weakly) fewer protection. A higher probability of contagion leads to (weakly) more attacks and (weakly) more protection.

## 9.4 Simulation

This subsection develops some simulations for the monopoly solution in the model of networked defenders under attack. Assume the cost of protection is:

$$C_U(\chi) = \frac{a}{\chi + b}. \tag{45}$$

In particular, $C_U(0) = a/b$ and $C_U(1) = a/(1+b)$. This cost function satisfies the following assumptions: $C_U(0) < +\infty$, $C_U(1) > 0$, and $dC_U/d\chi < 0$.

Consider three families of continuous, atomless distribution functions of gains and losses from a successful attack, $F_X$ and $F_S$. Each of the distributions has support on $\mathbb{R}_+$ with $F_X(0) = F_S(0) = 1$ and $F_X(+\infty) = F_S(+\infty) = 1$. Each of the distributions has a finite expected value. Finally, each of the distributions assigns lower probability to high than it assigns to low payoffs and losses from successful attacks. The three families of distribution functions are:

1. Lomax distributions[7] with CDFs $F_S(s) = 1 - \frac{\rho_S^2}{(s+\rho_S)^2}$ and $F_X(x) = 1 - \frac{\rho_X^2}{(x+\rho_X)^2}$,
2. Half-Normal distributions with variance parameters $\sigma_S$ and $\sigma_X$, folded at 0, with CDFs $F_X(x) = \mathrm{Erf}\left(\frac{x}{\sqrt{2}\sigma_X}\right)$ and $F_S(s) = \mathrm{Erf}\left(\frac{s}{\sqrt{2}\sigma_S}\right)$, and
3. exponential distributions with parameters $\theta_S$ and $\theta_X$, with CDFs $F_S(s) = 1 - e^{-\frac{1}{\theta_S}s}$ and $F_X(x) = 1 - e^{-\frac{1}{\theta_X}x}$.

In Table 1 we compile the probability distribution functions, the cumulative distribution functions and the expectations for the three distributions we are using in this subsection.

Table 1: Distribution functions used in the simulation

| Distribution | *pdf* | *cdf* | $E(\cdot)$ |
|---|---|---|---|
| Lomax | $f_X(x) = \frac{2\rho_X^2}{(x+\rho_X)^3}$<br>$f_S(s) = \frac{2\rho_S^2}{(s+\rho_S)^3}$ | $F_X(x) = 1 - \frac{\rho_X^2}{(x+\rho_X)^2}$<br>$F_S(s) = 1 - \frac{\rho_S^2}{(s+\rho_S)^2}$ | $E(x) = \rho_X$<br>$E(s) = \rho_S$ |
| Half-Normal | $f_X(x) = \frac{2}{\sqrt{2\pi}\sigma_X} e^{-\frac{x^2}{2\sigma_X^2}}$<br>$f_S(s) = \frac{2}{\sqrt{2\pi}\sigma_S} e^{-\frac{s^2}{2\sigma_S^2}}$ | $F_X(x) = \mathrm{Erf}\left(\frac{x}{\sqrt{2}\sigma_X}\right)$<br>$F_S(s) = \mathrm{Erf}\left(\frac{s}{\sqrt{2}\sigma_S}\right)$ | $E(x) = \sqrt{\frac{2}{\pi}}\sigma_X$<br>$E(s) = \sqrt{\frac{2}{\pi}}\sigma_S$ |
| Exponential | $f_X(x) = \frac{1}{\theta_X} e^{-\frac{1}{\theta_X}x}$<br>$f_S(s) = \frac{1}{\theta_S} e^{-\frac{1}{\theta_S}s}$ | $F_X(x) = 1 - e^{-\frac{1}{\theta_X}x}$<br>$F_S(s) = 1 - e^{-\frac{1}{\theta_S}s}$ | $E(x) = \theta_X$<br>$E(s) = \theta_S$ |

[7]The Lomax Distribution we use is one with a "shape" parameter equal to 2. It is also known as the Pareto Type II Distribution. The Lomax Distribution is related to the Beta Prime Distribution with parameters $\alpha = 1$ and $\beta = 2$, modulo a change of variables of the sort of $x = ky$, for some constant $k > 0$.

Figures 14-16 in the Appendix show how demand depends on the attackers' and defenders' distribution parameters. Given any price $p$, for the Lomax distributions, demand increases in $\rho_S$ and $\rho_X$, for the Half-Normal distributions, demand increases in $\sigma_S$ and $\sigma_X$, and for the exponential distributions, demand increases in $\theta_S$ and $\theta_X$.

Figures 4, 6, and 8 present demand, cost, profit and welfare for the Lomax distributions with $\rho_S = 1$ and $\rho_X = 0.5$, the Half-Normal distributions with $\sigma_S = 1$ and $\sigma_X = 0.5$, and the exponential distributions with $\theta_S = 2$ and $\theta_X = 1$, all assuming $\kappa = 0.5, \xi = 0.5, a = 0.1, b = 0.1$. Note that our model assumes that the cost to the attackers from an unsuccessful direct attack is normalized to 1.

The assumption on the distribution parameters is that the expected losses are larger than the expected gains from successful attacks. Indeed, it turns out that the industry is only viable if neither the expected losses nor the expected gains from attacks are not too low.

Figures 5, 7 and 9 display welfare and monopoly profit as functions of the share of protected defenders. Both welfare and profit have an internal maximum for our parameter constellation. Notice that the graphs visualize three solutions: The welfare maximum, the break-even point which represents the Bertrand equilibrium, and the monopoly solution. As expected, the share of protected defenders is largest in the welfare maximum, it is smaller in the Bertrand solution and smallest in the monopoly solution.

Figures 5, 7 and 9 illustrate that, for all three families of distributions, the welfare loss in the monopoly solution is sizeable, irrespective of whether the reference is the Bertrand solution or social optimum. They also illustrate that the Bertrand solution implies a welfare loss.

Figures 10 and 11 illustrate how the profit-maximizing share of protected defenders depends on the distributional parameters. The distribution of the defenders' losses enters only the monopolist's revenue, whereas that of the attackers' gains enters both the monopolist's revenue and its cost.

Figure 11 also illustrates how for the Half-Normal and exponential distributions, there may be multiple monopoly solutions. Assuming the expected gains are much higher than the expected losses from successful attacks and the Half-Normal distribution, Figures 12 and 13 similarly highlight that there may be several break-even points and several local (and even global) monopoly profit maxima.

# 10 Conclusion

This analysis provides two novel effects. First, there is a ***demand-side network effect***. This effect has a direct and an indirect channel. Suppose the proportion of protected defenders increases. Then the defenders' exposure decreases directly as contagion of perpetrated attacks is reduced. Furthermore this increase in the proportion of protected defenders creates a more adverse environment for attackers, discouraging some of them from attacking, hence also decreasing the defenders' exposure to an attack. Second, there is a **supply-side effect**. If fewer attackers attack, there are fewer modes of attack and the cost of individual protection decreases. We call these cost-reducing effects **economies of use**.

[*To be completed*]

# Appendix

**Proof of Proposition 1**

*Existence of equilibrium.* Consider the functions $g_+ : [0,1] \to [0,1]$ and $g_- : [0,1] \to [0,1]$, defined at every $\lambda \in [0,1]$, by:

$$g_+(\lambda) = \lambda, \tag{46}$$

and

$$g_-(\lambda) = F_S\left(\frac{P\left(F_X\left(\beta(\lambda)\right), \lambda\right)}{\delta\left(F_X(\beta(\lambda)), \lambda\right)}\right). \tag{47}$$

Suppose $g_-(0) = 0$. Then $g_-(0) = 0 = g_+(0)$. In this case, $(\chi^*, \lambda^*) = (F_X(\beta(0)), 0)$ is an equilibrium. Next suppose $g_-(1) = 1$. The, $g_-(1) = 1 = g_+(1)$. In this case, $(\chi^*, \lambda^*) = (F_X(\beta(1)), 1)$ is an equilibrium. Finally, suppose $g_-(0) > 0$ and $g_-(1) < 1$. Consider the function $g_-(\lambda) - g_+(\lambda)$. This function is continuous, strictly positive at $\lambda = 0$, strictly negative at $\lambda = 1$. By the intermediate value theorem, there exists a $\lambda^* \in ]0,1[$ such that $g_-(\lambda^*) = g_+(\lambda^*)$. In this case, $(\chi^*, \lambda^*) = (F_X(\beta(\lambda^*)), \lambda^*)$ is an equilibrium.

*Uniqueness of equilibrium.* Let $g(\lambda) = g_-(\lambda) - g_+(\lambda)$. If this function $g$ is strictly decreasing, the equilibrium has to be unique. This always holds if $dg_-/d\lambda < dg_+/d\lambda$, for all $0 < \lambda < 1$. Once more, to economize on notation, let $\chi = F_X(\beta(\lambda))$, for every $0 \leq \lambda \leq 1$. The derivative of $g_-$, at each point $\lambda$, is:

$$\frac{d}{d\lambda}[g_-] = \frac{f_S\left(\frac{P(\chi,\lambda)}{\delta(\chi,\lambda)}\right)}{\delta^2(\chi,\lambda)}\left\{\delta(\chi,\lambda)\frac{\partial P}{\partial \chi}\frac{d\chi}{d\lambda}(\lambda) + \delta(\chi,\lambda)\frac{\partial P}{\partial \chi} - P(\chi,\lambda)\frac{\partial \delta}{\partial \chi}\frac{d\chi}{d\lambda}(\lambda) - P(\chi,\lambda)\frac{\partial \delta}{\partial \lambda}\right\}.$$

Hence, dropping all arguments, the following inequalities are all equivalent:

$$\frac{d}{d\lambda}[g_-] < \frac{d}{d\lambda}[g_+],$$

$$\frac{f_S\left(\frac{P}{\delta}\right)}{\delta^2}\left(\delta\frac{\partial P}{\partial \chi}\frac{d\chi}{d\lambda} + \delta\frac{\partial P}{\partial \chi} - P\frac{\partial \delta}{\partial \chi}\frac{d\chi}{d\lambda} - P\frac{\partial \delta}{\partial \lambda}\right) < 1,$$

$$\delta\left(\frac{\partial P}{\partial \chi}\frac{\partial \chi}{\partial \lambda} + \frac{\partial P}{\partial \lambda}\right) < P\left(\frac{\partial \delta}{\partial \chi}\frac{\partial \chi}{\partial \lambda} + \frac{\partial \delta}{\partial \lambda}\right) + \frac{\delta^2}{f_S}.$$

Multiplying both sides by $\lambda$ and dividing by $\delta P$:

$$\frac{\partial P}{\partial \lambda}\frac{\lambda}{P} + \frac{\chi}{P}\frac{\partial P}{\partial \chi}\frac{d\chi}{d\lambda}\frac{\lambda}{\chi} < \frac{\partial \delta}{\partial \lambda}\frac{\lambda}{\delta} + \frac{\chi}{\delta}\frac{\partial \delta}{\partial \chi}\frac{d\chi}{d\lambda}\frac{\lambda}{\chi} + \frac{\delta\lambda}{P f_S}.$$

Using the definitions of the elasticities and $\overline{\varepsilon}$ leads to:

$$\varepsilon_{P,\lambda} + \varepsilon_{P,\chi}\varepsilon_{\chi,\lambda} < \varepsilon_{\delta,\lambda} + \varepsilon_{\delta,\chi}\varepsilon_{\chi,\lambda} + \overline{\varepsilon}.$$

As $\overline{\varepsilon} \geq 0$, a sufficient condition for uniqueness is $\varepsilon_{P,\lambda} + \varepsilon_{P,\chi}\varepsilon_{\chi,\lambda} < \varepsilon_{\delta,\lambda} + \varepsilon_{\delta,\chi}\varepsilon_{\chi,\lambda}$. This completes the proof. ■

**Proof of Corollary 1**

**Part 1:** suppose $F_{\widehat{S}}(s) < F_S(s)$, for all $s > 0$, and $X$ is unchanged. Define the functions $g_+ : [0,1] \to [0,1]$ and $g_- : [0,1] \to [0,1]$ as in the proof of Proposition 1. Define function $\hat{g}_- : [0,1] \to [0,1]$ similarly as $g_- : [0,1] \to [0,1]$, just changing the cumulative distribution $S$ by $\widehat{S}$. Then, for all $0 < \lambda < 1$:

$$\hat{g}_-(\lambda) = F_{\widehat{S}}\left(\frac{P\left(F_X\left(\beta(\lambda)\right),\lambda\right)}{\delta\left(F_X(\beta(\lambda)),\lambda\right)}\right) < F_S\left(\frac{P\left(F_X\left(\beta(\lambda)\right),\lambda\right)}{\delta\left(F_X(\beta(\lambda)),\lambda\right)}\right) = g_-(\lambda).$$

The unique point of the domain where functions $\hat{g}_- : [0,1] \to [0,1]$ and $g_+ : [0,1] \to [0,1]$ intersect each other is $\lambda = \widehat{\lambda}$. As $\hat{g}_-(\lambda^*) < g_-(\lambda^*)$, then $\hat{g}_-(\lambda^*) - g_+(\lambda^*) < g_-(\lambda^*) - g_+(\lambda^*) = 0$. As, by assumption, the function $\hat{g}_-(\lambda) - g_+(\lambda)$ is decreasing and $\hat{g}_-(\widehat{\lambda}) - g_+(\widehat{\lambda}) = 0$, then $\widehat{\lambda} < \lambda^*$.

Define $\widehat{\chi} = F_X\left(\beta(\widehat{\lambda})\right)$. As $\widehat{\lambda} < \lambda^*$, function $\beta$ is decreasing and $F_X$ is increasing, then $\widehat{\chi} = F_X\left(\beta(\widehat{\lambda})\right) > F_X\left(\beta(\lambda^*)\right) = \chi^*$.

**Part 2:** suppose $F_{\widehat{\widehat{X}}}(x) < F_X(x)$, for all $x > 0$, and $S$ is unchanged. Define the functions $g_+ : [0,1] \to [0,1]$ and $g_- : [0,1] \to [0,1]$ as in the proof of Proposition 1. Define function $\widehat{\widehat{g}}_- : [0,1] \to [0,1]$ similarly as $g_- : [0,1] \to [0,1]$, just changing the cumulative distribution $X$ by $\widehat{\widehat{X}}$. Then, for all $0 < \lambda < 1$:

$$\widehat{\widehat{g}}_-(\lambda) = F_S\left(\frac{P\left(F_{\widehat{\widehat{X}}}\left(\beta(\lambda)\right),\lambda\right)}{\delta\left(F_{\widehat{\widehat{X}}}(\beta(\lambda)),\lambda\right)}\right) < F_S\left(\frac{P\left(F_X\left(\beta(\lambda)\right),\lambda\right)}{\delta\left(F_X(\beta(\lambda)),\lambda\right)}\right) = g_-(\lambda).$$

The unique point of the domain where functions $\widehat{\widehat{g}}_- : [0,1] \to [0,1]$ and $g_+ : [0,1] \to [0,1]$ intersect each other is $\lambda = \widehat{\widehat{\lambda}}$. As $\widehat{\widehat{g}}_-(\lambda^*) < g_-(\lambda^*)$, then $\widehat{\widehat{g}}_-(\lambda^*) - g_+(\lambda^*) < g_-(\lambda^*) - g_+(\lambda^*) = 0$. As, by assumption, the function $\widehat{\widehat{g}}_-(\lambda) - g_+(\lambda)$ is decreasing and $\widehat{\widehat{g}}_-(\widehat{\widehat{\lambda}}) - g_+(\widehat{\widehat{\lambda}}) = 0$, then $\widehat{\widehat{\lambda}} < \lambda^*$.

To be completed

**Proof of Lemma 1**

Equation (8) defines $\lambda^*$ as an implicit function of $p$. Using the implicit function theorem, taking the implicit derivative with respect to $p$ in both sides of equation (8) and isolating $d\lambda^*/dp$ leads to equation (9), where the derivatives $\partial\delta/\partial\chi$ and $\partial\delta/\partial\lambda$ are calculated at $(\chi^*, \lambda^*) = (F_X(\beta(\lambda^*)), \lambda^*)$, and the derivative $d\beta/d\lambda$ is calculated at $\lambda^*$. This completes the first part of the proof.

For the second part, note that, by equation (7), using the chain rule:

$$\frac{d}{d\lambda}[\chi^*] = f_X(\beta(\lambda^*))\frac{d}{d\lambda}[\beta]. \tag{48}$$

Using equation (9) and (48), we find the share $\chi^*$ of non-attacking attackers reacts to changes in the price $p$ of protection as given in equation (10). This completes the proof. ■

**Proof of Lemma 2**

Both the numerator and the denominator of the right-hand side of equation (9) are always positive. This completes the proof of the first part. By the chain rule:

$$\frac{d\chi^*}{dp} = \frac{d\chi^*}{d\lambda}\frac{d\lambda^*}{dp}. \tag{49}$$

Substitute $\frac{d\chi^*}{d\lambda}$ by the right-hand side of equation (48) to get

$$\frac{d\chi^*}{dp} = f_X(\beta(\lambda^*))\frac{d}{d\lambda}[\beta]\frac{d\lambda^*}{dp}.$$

By the first part of this proof, Assumption 4 and $f_X > 0$, this expression is negative. This completes the proof. ■

**Proof of Proposition 2**

To be written up

**Proof of Lemma 3**

Because $\frac{C_D}{1-\lambda^*} > 0$ and $-\left((1-\lambda^*)\frac{dC_U}{d\chi} + \frac{dC_D}{d\chi}\right)\frac{d}{d\lambda}[\chi^*] < 0$, then:

$$AC = C_U + \frac{C_D}{1-\lambda^*} > C_U > C_U - \left((1-\lambda^*)\frac{dC_U}{d\chi} + \frac{dC_D}{d\chi}\right)\frac{d}{d\lambda}[\chi^*] = MC.$$

This proves that the inequalities in (14) must hold. This completes the proof. ■

**Proof of Lemma 4**

Taking the derivative of the average cost with respect to the price leads to:

$$\begin{aligned}
\frac{dAC}{dp} &= \frac{d}{dp}\left[C_U + \frac{C_D}{1-\lambda^*}\right] \\
&= \frac{d\lambda^*/dp}{1-\lambda^*}\left(\frac{C_D}{1-\lambda^*} + \left((1-\lambda^*)\frac{d}{d\chi}(C_U] + \frac{d}{d\chi}[C_D]\right)\frac{d\chi^*}{d\lambda}\right) \\
&= \frac{d\lambda^*/dp}{1-\lambda^*}\left(\frac{C_D}{1-\lambda^*} + C_U - MC\right) \\
&= \frac{d\lambda^*/dp}{1-\lambda^*}\left(AC - MC\right) \\
&= \frac{-\varepsilon_{Q^*,p}}{p}(AC - MC).
\end{aligned}$$

As $-\varepsilon_{Q^*,p} > 0$, $p > 0$ and, by the previous lemma, $AC - MC > 0$. Hence, $\frac{dAC}{dp} > 0$. This completes the proof. ■

**Proof of Corollary 2**

As $dAC/dp > 0$ and $d\lambda^*/dp > 0$, then:

$$\frac{dAC}{d(1-\lambda)} = -\frac{dAC}{d\lambda} = -\frac{dAC}{d\lambda} = \frac{-dAC/dp}{d\lambda^*/dp} < 0.$$

This completes the argument. ■

**Proof of Proposition 3**

Consider the first derivative of the monopoly profit:

$$\frac{d\pi}{dp} = (1-\lambda^*) - \left(p\frac{d}{dp}[\lambda^*] + \left((1-\lambda^*)\frac{\partial C_U}{\partial \chi} + \frac{\partial C_D}{\partial \chi}\right)\frac{d}{dp}[\chi^*] - C_U(\chi^*)\frac{d}{dp}[\lambda^*]\right).$$

Taking the limit of the marginal profit as $p \to +\infty$ results in:

$$\lim_{p\to+\infty}\frac{d\pi}{dp} = \lim_{p\to+\infty}\left((1-\lambda^*) - \left(p\frac{d}{dp}[\lambda^*] + \left((1-\lambda^*)\frac{\partial C_U}{\partial \chi} + \frac{\partial C_D}{\partial \chi}\right)\frac{d}{dp}[\chi^*] - C_U(\chi^*)\frac{d}{dp}[\lambda^*]\right)\right).$$

The monopolist charges a finite price if $\lim_{p\to+\infty}\frac{d\pi}{dp} < 0$. This condition holds if

$$\lim_{p\to+\infty}\frac{1-\lambda^*}{\frac{d}{dp}[\lambda^*]} < \lim_{p\to+\infty}\left(p - C_U + \left((1-\lambda^*)\frac{\partial C_U}{\partial \chi} + \frac{\partial C_D}{\partial \chi}\right)\frac{\partial \chi}{\partial \lambda}\right). \tag{50}$$

Consider the left-hand side of of inequality (50). Using equation (9):

$$\lim_{p\to+\infty}\frac{1-\lambda^*}{\frac{d}{dp}[\lambda^*]} = \lim_{p\to+\infty}\frac{1-F_S\left(\frac{p}{\delta}\right)}{f_S\left(\frac{p}{\delta}\right)}\left(\delta + \frac{p}{\delta}f_S\left(\frac{p}{\delta}\right)\left(\delta_\chi\frac{\partial \chi^*}{\partial \lambda} + \delta_\lambda\right)\right).$$

The exposure, $\delta$, approaches 1 as $p \to +\infty$. Under Assumption 10, $\int_0^{+\infty} p f_S(p) dp < +\infty$ and therefore $\lim_{p\to+\infty} p f_S\,(p) = 0$. Because under Assumption 10, for all $p \geq \overline{p}$, $f_S$ is weakly decreasing, $p f_S(p) \leq p f_S\,(p/\delta)$ and $\lim_{p\to+\infty}(p/\delta) f_S\,(p/\delta) = 0$. Using Assumption 8,

$$\lim_{p\to+\infty} \frac{1-\lambda^*}{\frac{d}{dp}[\lambda^*]} = \lim_{p\to+\infty} \frac{1-F_S\,(p/\delta)}{f_S(p/\delta)}. \tag{51}$$

Now, consider the right-hand-side of inequality (50). Because

$$\frac{\partial \chi^*}{\partial \lambda} = f_X\,(\beta)\,\frac{d}{d\lambda}\,[\beta] < 0 \qquad \text{and} \qquad (1-\lambda^*)\frac{\partial C_U}{\partial \chi} + \frac{\partial C_D}{\partial \chi} \leq 0,$$

then:

$$\lim_{p\to+\infty}\,(p - C_U) \leq \lim_{p\to+\infty} \left(p - C_U + \left((1-\lambda^*)\frac{\partial C_U}{\partial \chi} + \frac{\partial C_D}{\partial \chi}\right)\frac{\partial \chi}{\partial \lambda}\right).$$

In order to establish inequality (50), it suffices to prove that:

$$\lim_{p\to+\infty} \frac{1-\lambda^*}{\frac{d}{dp}[\lambda^*]} < \lim_{p\to+\infty}\,(p - C_U)\,.$$

Given equation (51), this last inequality is equivalent to:

$$\lim_{p\to+\infty} \frac{1-F_S\left(\frac{p}{\delta}\right)}{f_S\left(\frac{p}{\delta}\right)} < \lim_{p\to+\infty}\,(p - C_U)\,.$$

This last inequality is equivalent to Assumption 9, $\lim_{p\to+\infty}(p - C_U)\frac{f_S(p/\delta)}{1-F_S(p/\delta)} > 1$. Hence, inequality (50) holds, and the monopoly price is finite. This completes the proof. ■

**Proof of Proposition 4**

This result summarizes Lemma 4 and Equation (21). This completes the proof. ■

**Proof of Proposition 5**

The defenders' welfare is given by equation (26). Hence, the derivative of $W_J$, with respect to the price is

$$\frac{dW_J}{dp} = -1 + p\frac{d}{dp}[\lambda^*] + \lambda^* - \delta F_S^{-1}(\lambda^*)\frac{d}{dp}[\lambda^*] - \delta_p \int_0^{F_S^{-1}(\lambda *)} s_j f_S(s_j) ds_j.$$

Using Leibnitz' rule and the fact that, in equilibrium, $F_S^{-1}(\lambda^*) = p/\delta(\chi^*)$, the derivative of $W_J$, with respect to the price simplifies to:

$$\frac{dW_J}{dp} = -(1-\lambda^*) - \delta_p \int_0^{F_S^{-1}(\lambda *)} s_j f_S(s_j) ds_j, \tag{52}$$

which is strictly negative. This completes the proof. ■

**Proof of Proposition** 6

Under assumptions 8 – 10, there is a finite monopoly price $p = p^M$, at which $d\pi/dp = 0$. At this price, $\frac{d}{dp}W\Big|_{p=p^M} = \frac{d}{dp}W_J\Big|_{p=p^M}$. Hence, Proposition 5 implies the result. This completes the proof. ■

**Proof of Proposition 7**

Use equation (27) to find the derivative of $W$ with respect to the price:

$$\begin{aligned}\frac{d}{dp}[W] &= \overbrace{-\delta F_S^{-1}(\lambda^*)\frac{d}{dp}[\lambda^*]}^{=-p\frac{d}{dp}[\lambda^*]} - \delta_p \int_{s_j=0}^{F_S^{-1}(\lambda^*)} s_j f_S(s_j) ds_j + C_U \frac{d}{dp}[\lambda^*] - (1-\lambda^*)\frac{dC_U}{dp} - \frac{dC_D}{dp} \\ &= -\delta_p \int_{s_j=0}^{F_S^{-1}(\lambda^*)} s_j f_S(s_j) ds_j - (p - C_U)\frac{d}{dp}[\lambda^*] - (1-\lambda^*)\frac{dC_U}{dp} - \frac{dC_D}{dp}.\end{aligned}$$

Hence, for all $p \geq \underline{p}$, $\frac{d}{dp}[W] < 0$. This completes the proof. ■

**Proof of Proposition 8**

The first derivative of the welfare function with respect to the price is:

$$\frac{d}{dp}[W] = -\delta_p \int_0^{F_S^{-1}(\lambda^*)} s_j f_S(s_j) ds_j - (p - C_U)\frac{d}{dp}[\lambda^*] - (1-\lambda^*)\frac{\partial C_U}{\partial p} - \frac{\partial C_D}{\partial p}.$$

Take the limit with $p \to 0$. Then the first term goes to zero (using Assumption 8) and

$$\lim_{p\to 0}\frac{d}{dp}[W] = \lim_{p\to 0}\left\{\left(C_U(\chi^*) - (1-\lambda^*)\frac{\partial C_U}{\partial \chi}\frac{d}{d\lambda}[\chi^*] - \frac{\partial C_D}{\partial \chi}\frac{d}{d\lambda}[\chi^*]\right)\frac{d}{dp}[\lambda^*]\right\}.$$

Under Assumption 8, $\lim_{p\to 0}\frac{d}{dp}[\lambda^*] > 0$. Therefore, and using that, for $p \to 0$, $\lambda^* \to 0$ and $\chi^* \to 1$, this expression is positive if

$$C_U(1) > \left(\frac{\partial C_U(1)}{\partial \chi} + \frac{\partial C_D(1)}{\partial \chi}\right)\frac{d}{d\lambda}[\chi^*(0)].$$

This completes the proof. ■

**Proof of Proposition 9**

# To be written up

**Proof of Proposition 10**

**(1) Existence** Using equations 8 and 7, we rewrite $p = C_U(\chi^*(p))$ as:

$$F_S^{-1}\left(\beta^{-1}\left(F_X^{-1}(\chi^*(p))\right)\right)\delta\left(\chi^*(p), \beta^{-1}\left(F_X^{-1}(\chi^*(p))\right)\right) = C_U(\chi^*(p)).$$

Because Lemma 2 establishes that $\chi^*(p)$ is strictly increasing in $p$, to show there exists a $p$ such that $p = C_U(\chi^*(p))$, it suffices to show that there exists a $\chi$ such that

$$C_U(\chi) = F_S^{-1}\left(\beta^{-1}\left(F_X^{-1}(\chi)\right)\right)\delta\left(\chi, \beta^{-1}\left(F_X^{-1}(\chi)\right)\right). \tag{53}$$

Consider $\chi = 0$. We have $F_X^{-1}(0) = 0$, $\beta^{-1}(0) \to +\infty$, $F_S^{-1}(+\infty) \to +\infty$ and $C_U(0) < +\infty$. Because $\delta(0, \lambda) > 0$, then $C_U(0) < F_S^{-1}\left(\beta^{-1}\left(F_X^{-1}(0)\right)\right)\delta\left(0, \beta^{-1}\left(F_X^{-1}(0)\right)\right)$. Next, consider $\chi = 1$. We have $F_X^{-1}(1) \to +\infty$, $\beta^{-1}(+\infty) = 0$, $F_S^{-1}(0) = 0$ and $C_U(1) > 0$. Because $\delta(1, \lambda) < +\infty$, then $C_U(1) > F_S^{-1}\left(\beta^{-1}\left(F_X^{-1}(1)\right)\right)\delta\left(1, \beta^{-1}\left(F_X^{-1}(1)\right)\right)$. Both sides of this equation are smooth functions in $\chi$. Hence, the intermediate value theorem implies that there exists a $\chi$ such that equation (53) holds. At the smallest price for which $p = C_U(\chi^*(p))$, no firm has a unilateral incentive to deviate: undercutting would attract the whole demand but at a loss for every unit; pricing higher would yield a revenue of zero.

**(2) Uniqueness** Let $\frac{dC_U}{dp}[C_U(\chi^*(\underline{p}))] < 1$. Then, for any $\widetilde{p} > \underline{p}$ such that $\widetilde{p} = C_U(\chi^*(\widetilde{p}))$, there exists a price $\underline{p} < p < \widetilde{p}$ which would attract the entire demand and yields a strictly positive profit. Let $\underline{p}$ be the unique equilibrium price. Suppose to the contrary that $\frac{dC_U}{dp}[C_U(\chi^*(\underline{p}))] \geq 1$. As $C_U(\chi^*(0)) > 0$, we must have $\frac{dC_U}{dp}[C_U(\chi^*(\underline{p}))] = 1$. Furthermore, because $\lim_{p\to+\infty} C_U(\chi^*(p)) < +\infty$, there must be at least one more $p > \underline{p}$ such that $p = C_U(\chi^*(p))$. Denote $\widehat{p} = \min\left\{p > \underline{p} \middle| p = C_U(\chi^*(p))\right\}$. Then there is no price $p < \widehat{p}$ which yields a strictly positive profit and, hence, $\widehat{p}$ is an equilibrium, contradicting uniqueness. This completes the proof. ■

**Proof of Corollary 3**

Because $C_U(\chi)\varepsilon_{C_U,\chi} > p\varepsilon_{p,\chi}$ holds for all $\chi$, then $\frac{d}{d\chi}\left[C_U\left(\chi^*\left(\underline{p}\right)\right)\right]\frac{d}{dp}\left[\chi^*\left(\underline{p}\right)\right] < 1$. Hence, uniqueness holds. This completes the proof. ■

**Proof of Corollary 4**

To be written up

**Proof of Corollary 5**

To be written up

**Proof of Corollary 6**

This result follows from Proposition 1. This completes the proof. ■

**Proof of Proposition 11**

**Part (*i*)**. **to be adjusted/checked/rewritten completely** – For each $p > 0$, define the function $x \mapsto g_-(\lambda; p)$, as before, by equation (47). Because $F_X$ is weakly increasing and because $f_S(\cdot)(1-\kappa\chi)/(1-\chi)(\kappa\xi - 1/(F_S(\cdot))^2) < 0$ and hence, $\kappa\xi F_S(p(1-\kappa\chi)/(1-\chi)) - [1+\kappa\xi] + 1/F_S(p(1-\kappa\chi)/(1-\chi))$ is decreasing in $p$, we have that $C > C'$ implies $g_-(\chi; C) \leq g_-(\chi; C')$. Function $g_+$ does not depend on parameter $C$ and, hence, an increase in $C$ weakly decreases the equilibrium value for $\chi^*$. Because $F_X$ is weakly increasing, by equation (1), a weak decrease in $\chi^*$ generates a weak decrease in $\kappa\lambda^* - [1+\kappa] + 1/\lambda^*$. Therefore, because $\kappa - 1/(\lambda^*)^2 < 0$, $\lambda^*$ must be weakly increasing.

**Part (*ii*)**. For each $\kappa > 0$, define the function $x \mapsto g_-(\chi; \kappa)$, as before, by equation (47).

As $F_X$ is weakly increasing and because $\xi F_S(\cdot) - \xi - \kappa\xi f_S(\cdot)C\chi/(1-\chi) - 1/(F_S(\cdot))^2 f_S(\cdot)C\chi/(1-\chi) < 0$ and therefore, $\kappa\xi F_S(C(1-\kappa\chi)/(1-\chi)) - [1+\kappa\xi] + 1/F_S(C(1-\kappa\chi)/(1-\chi))$ is decreasing in $\kappa$, we have that $\kappa > \kappa'$ implies $g_-(\chi; \kappa) \leq g_-(\chi; \kappa')$. Function $g_+$ does not depend on parameter $\kappa$ and, hence, an increase in $\kappa$ weakly decreases the equilibrium value for $\chi^*$. By equation (2), the equilibrium value for $\lambda^*$ weakly decreases. This completes the proof. ■

# A Figures

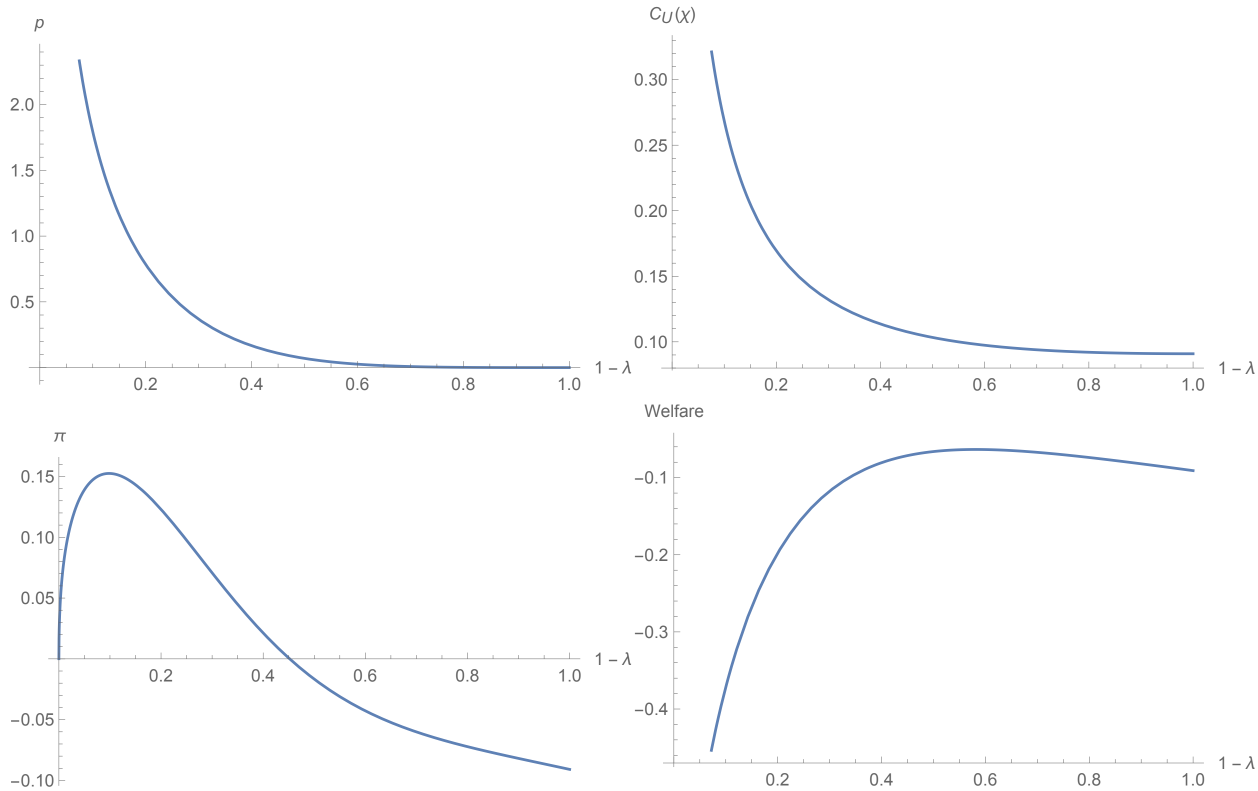


Figure 4: Willingness to pay, cost, profit and welfare for the Lomax distributions as functions of the share of protected defenders, $1 - \lambda$. (Assumptions: $\kappa = 0.5, \xi = 0.5, a = 0.1, b = 0.1, E[s] = 1, E[x] = 0.5$. The cost of a failed direct attack has been normalised to 1.)

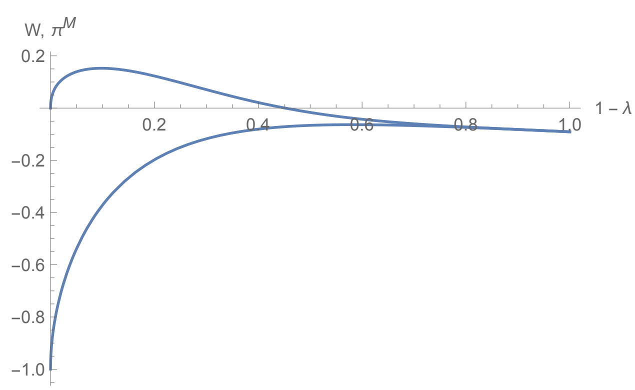


Figure 5: Profit and welfare for the Lomax distributions as functions of the share of protected defenders, $1 - \lambda$. The welfare maximizing share of unprotected defenders is smaller than the break-even share and the monopoly profit-maximizing share. (Assumptions: $\kappa = 0.5, \xi = 0.5, a = 0.1, b = 0.1, E[s] = 1, E[x] = 0.5$. The cost of a failed direct attack has been normalised to 1.)

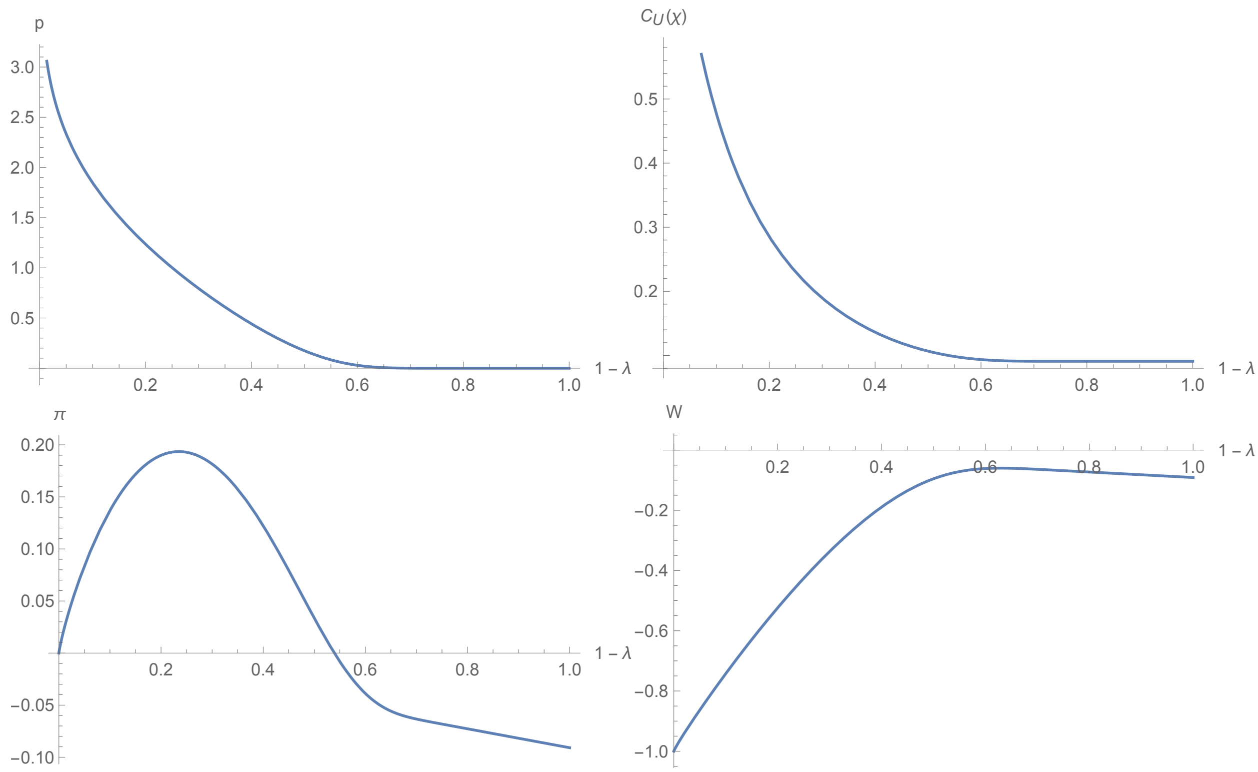


Figure 6: Willingness to pay, cost, profit and welfare for the Half-Normal distributions as functions of the share of protected defenders, $1-\lambda$. (Assumptions: $\kappa = 0.5, \xi = 0.5, a = 0.1, b = 0.1, E[s] = 1, E[x] = 0.5$. The cost of a failed direct attack has been normalised to 1.)

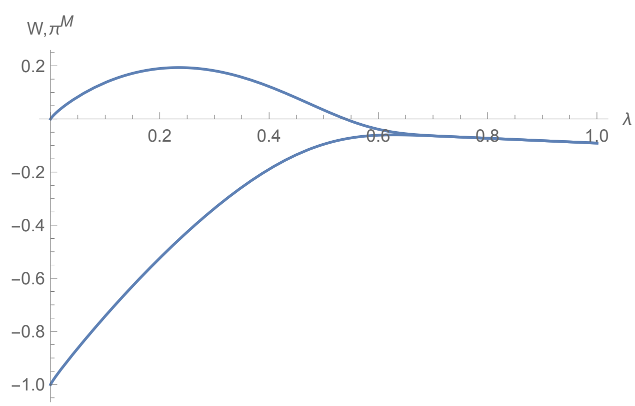


Figure 7: Profit and welfare for the Half-Normal distributions as functions of the share of protected defenders, $1-\lambda$. The welfare maximizing share of unprotected defenders is smaller than the break-even share and the monopoly profit-maximizing share. (Assumptions: $\kappa = 0.5, \xi = 0.5, a = 0.1, b = 0.1, E[s] = 1, E[x] = 0.5$. The cost of a failed direct attack has been normalised to 1.)

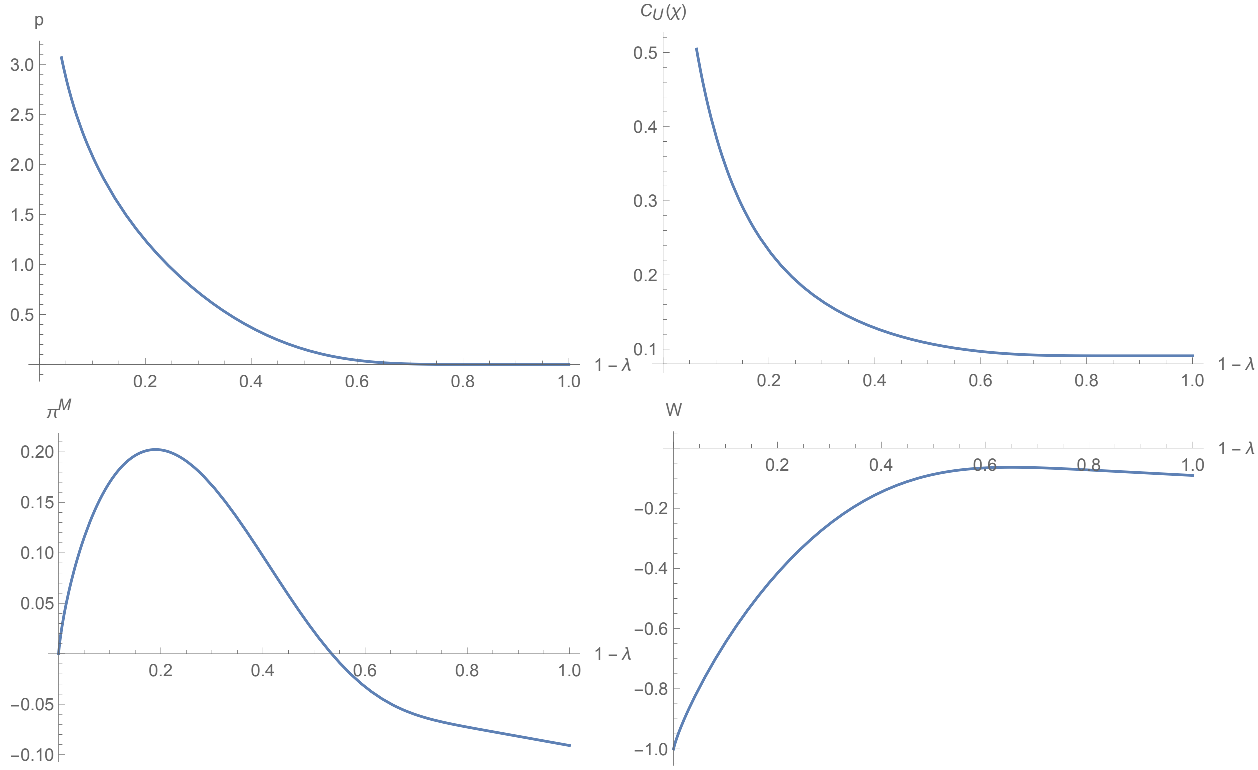


Figure 8: Willingness to pay, cost, profit and welfare for the exponential distributions as functions of the share of protected defenders, $1-\lambda$. We assume $\kappa = 0.5, \xi = 0.5, a = 0.1, b = 0.1, E[s] = 1, E[x] = 0.5$. The cost of a failed direct attack has been normalised to 1.

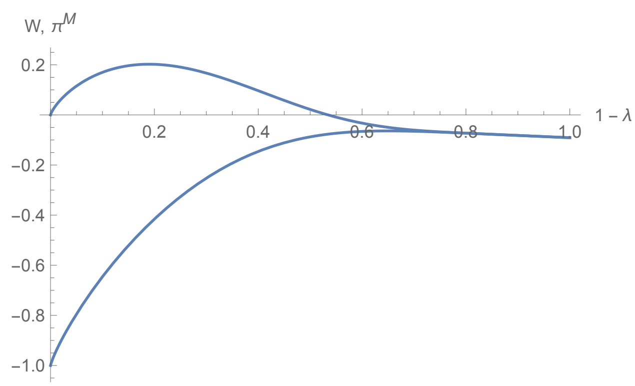


Figure 9: Profit and welfare for the exponential distributions as functions of the share of protected defenders, $1-\lambda$. The welfare maximizing share of unprotected defenders is smaller than the break-even share and the monopoly profit-maximizing share. (Assumptions: $\kappa = 0.5, \xi = 0.5, a = 0.1, b = 0.1, E[s] = 1, E[x] = 0.5$. The cost of a failed direct attack has been normalised to 1.)

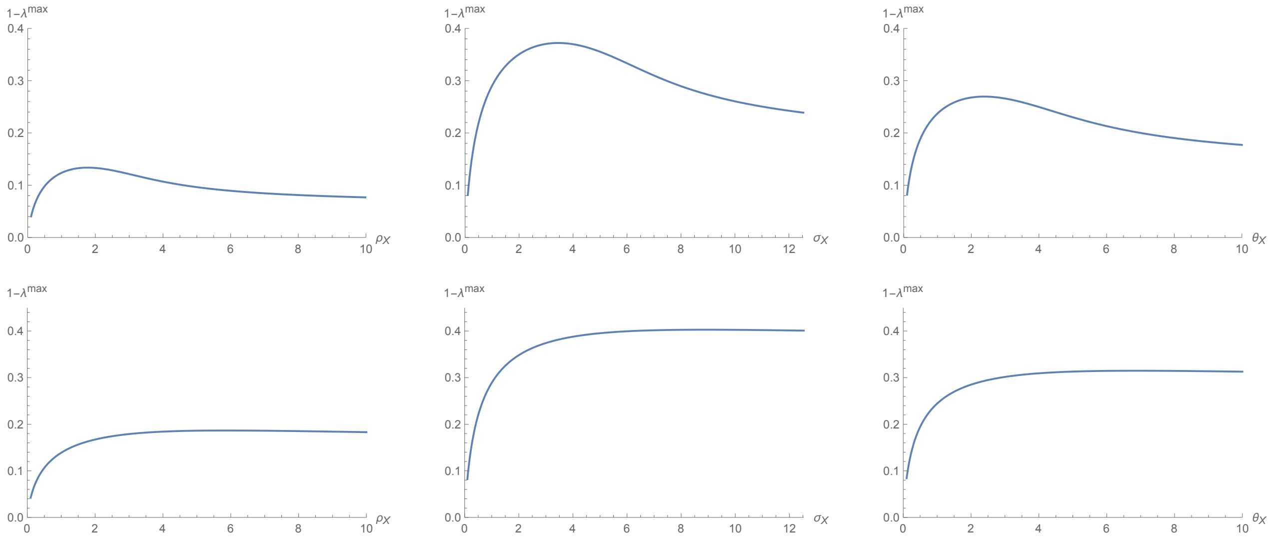


Figure 10: Profit-maximizing share of protected defenders as a function of the attackers' distribution parameters. Left: Lomax Distribution. Center: Half-Normal distribution. Right: exponential Distribution. (Assumptions: $\kappa = 0.5, \xi = 0.5, a = 0.1, b = 0.1$. Top: $E[s] = 1$. Bottom: $E[s] = 4$. The cost of a failed direct attack has been normalised to 1.)

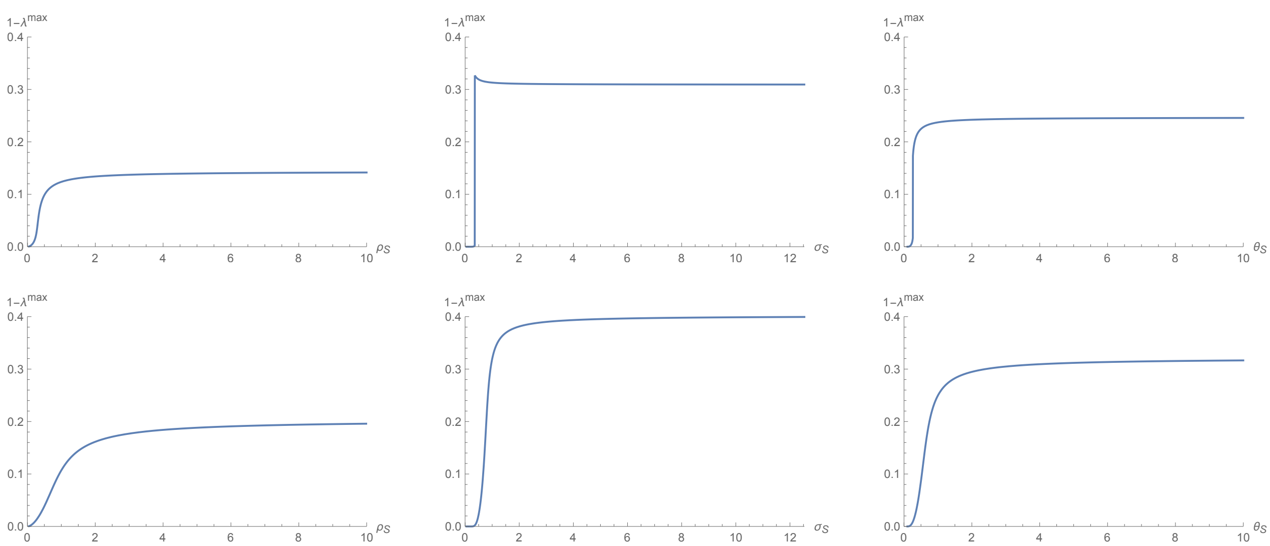


Figure 11: Profit-maximizing share of protected defenders as a function of the defenders' distribution parameters. Left: Lomax Distribution. Center: Half-Normal distribution. Right: exponential Distribution. (Assumptions: $\kappa = 0.5, \xi = 0.5, a = 0.1, b = 0.1$. Top: $E[x] = 1$. Bottom: $E[x] = 4$. The cost of a failed direct attack has been normalised to 1.)

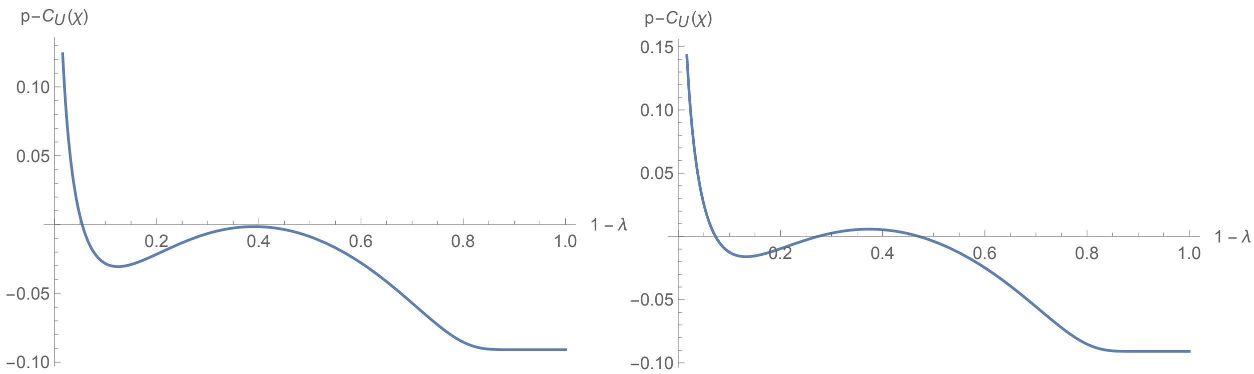


Figure 12: With the Half-Normal distribution, assuming $\kappa = 0.5, \xi = 0.5, a = 0.1, b = 0.1$, and $\sigma_S = 0.45, \sigma_X = 2$ (left; $E[s] \approx .577, E[x] \approx 2.51$) and $\sigma_S = 0.46, \sigma_X = 2$ (right; $E[s] \approx .564, E[x] \approx 2.51$), there are multiple break-even points. Right: Only the lowest price (highest $1 - \lambda$) break-even point is a Bertrand equilibrium. Left: Both break-even points are Bertrand equilibria.

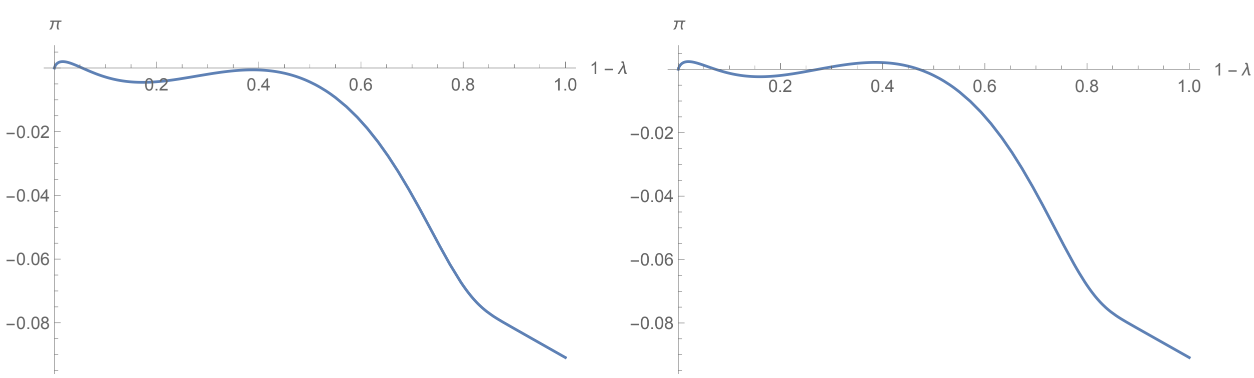


Figure 13: With the Half-Normal distribution, assuming $\kappa = 0.5, \xi = 0.5, a = 0.1, b = 0.1$, and $\sigma_S = 0.45, \sigma_X = 2$ (left; $E[s] \approx .577, E[x] \approx 2.51$) and $\sigma_S = 0.46, \sigma_X = 2$ (right; $E[s] \approx .564, E[x] \approx 2.51$), there are multiple local monopoly profit maxima.

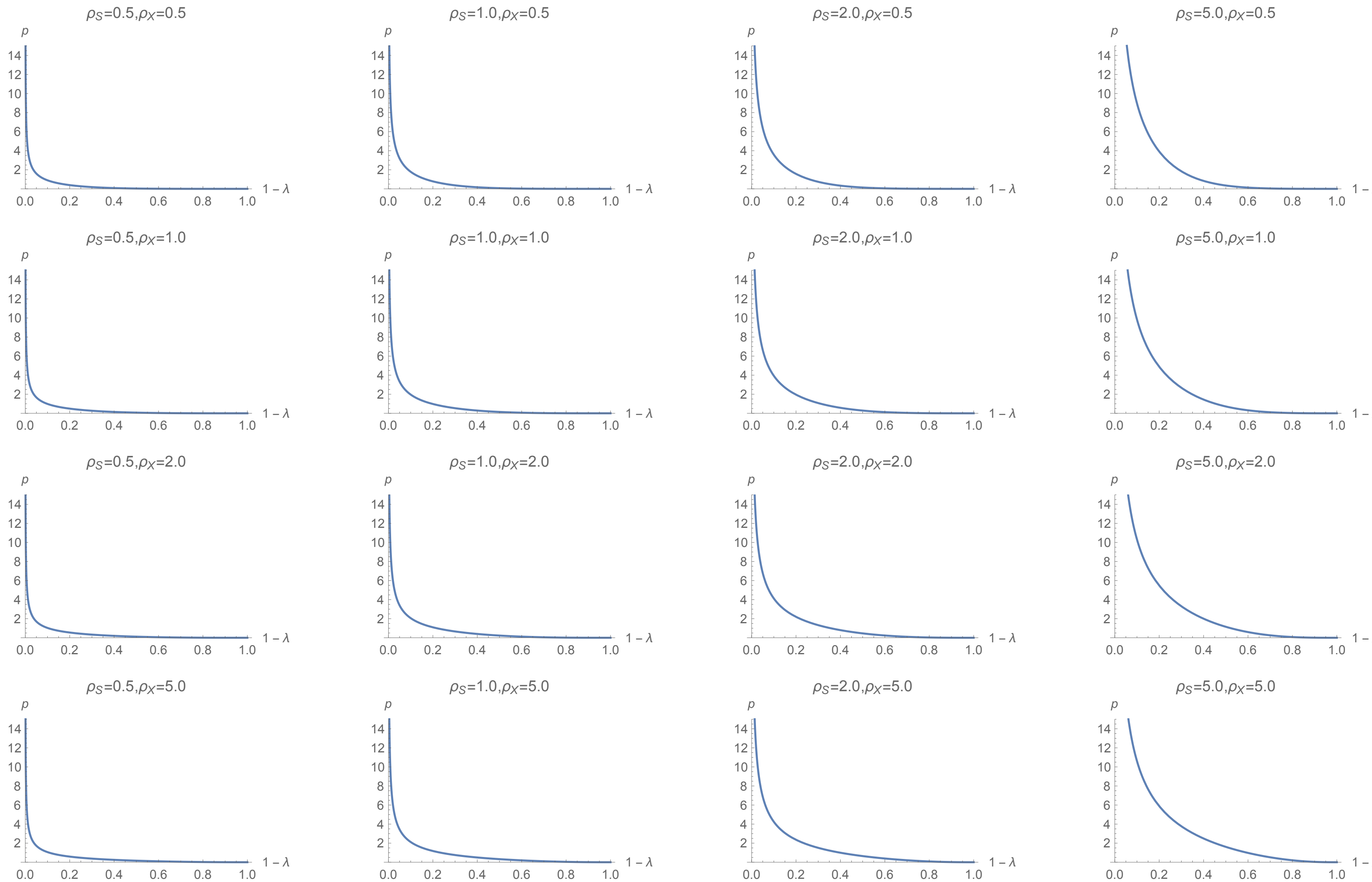


Figure 14: Demand functions for Lomax distribution for several values of $\rho_S$ and $\rho_X$. We assume $\kappa = 0.5, \xi = 0.5, a = 0.1, b = 0.1$. The cost of a failed direct attack has been normalised to 1.

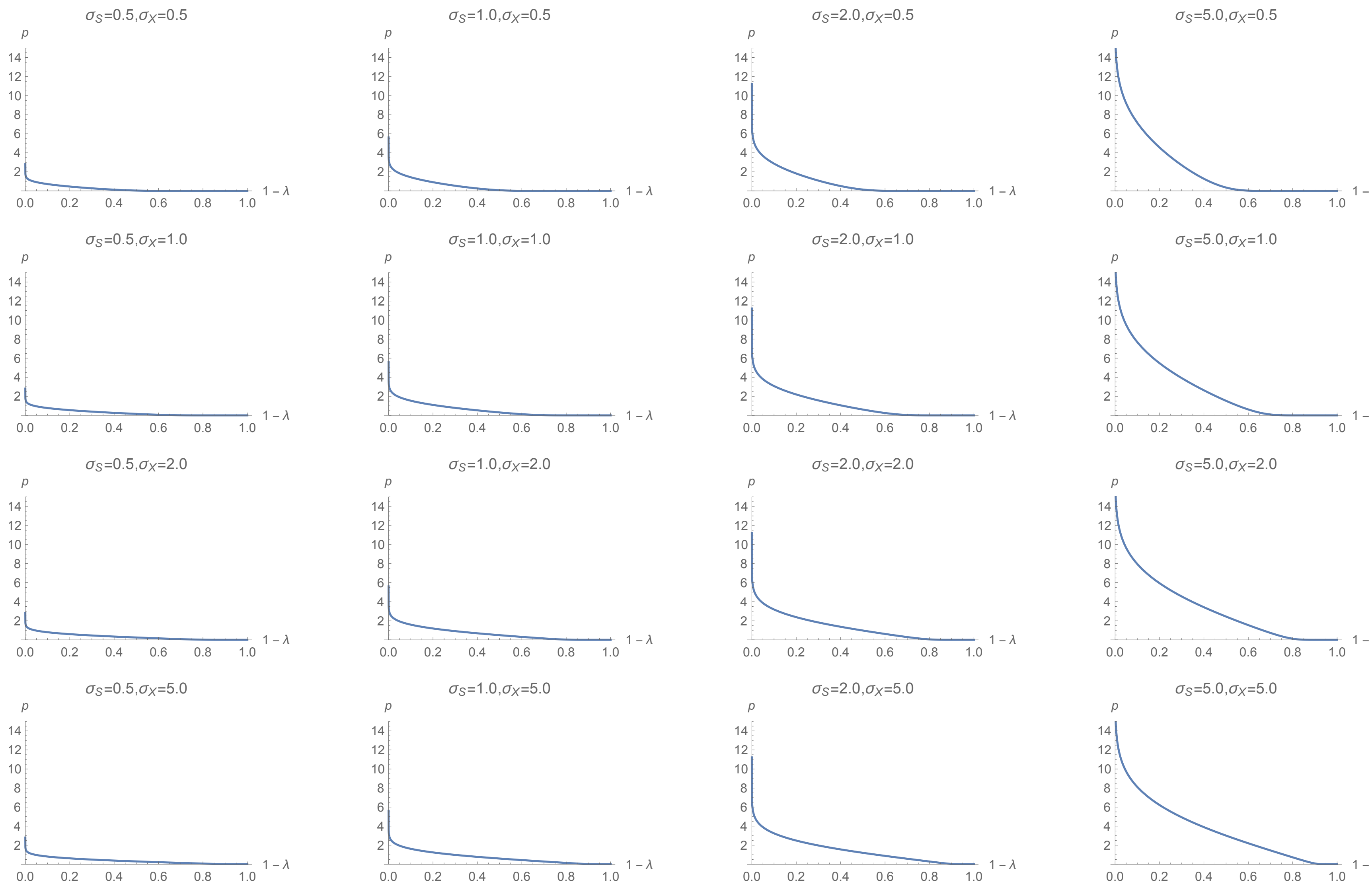


Figure 15: Demand functions for Half-Normal distribution for several values of $\sigma_S$ and $\sigma_X$. We assume $\kappa = 0.5, \xi = 0.5, a = 0.1, b = 0.1$. The cost of a failed direct attack has been normalised to 1.

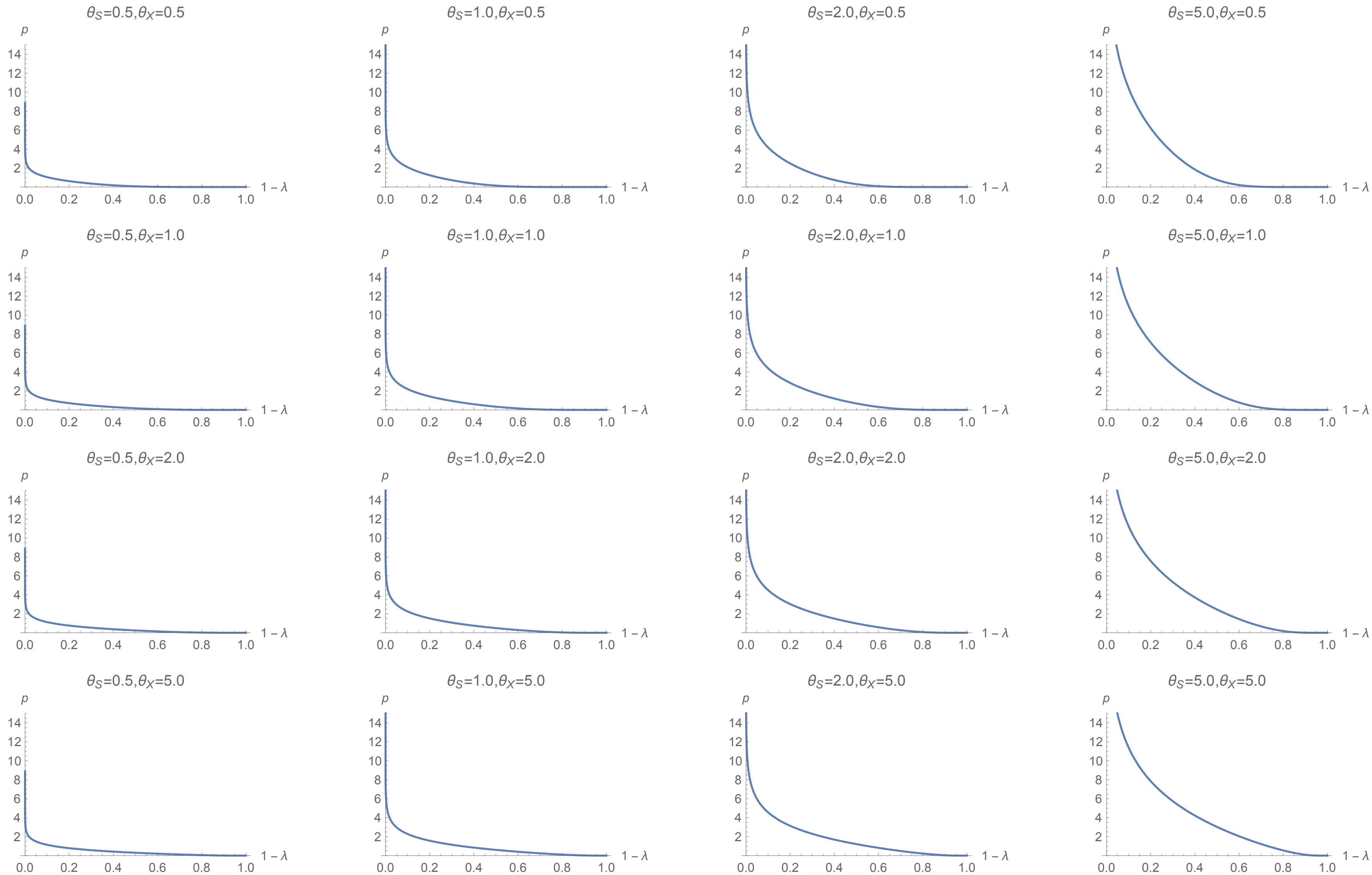


Figure 16: Demand functions for exponential distribution for several values of $\theta_S$ and $\theta_X$. We assume $\kappa = 0.5, \xi = 0.5, a = 0.1, b = 0.1$. The cost of a failed direct attack has been normalised to 1.

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