Information and Market Price Manipulation in the Unique Equilibrium of a Sequential Trade Model – Preliminary and Incomplete –

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Abstract. In asymmetric information models of financial markets, trading behavior imperfectly reveals the private information held by traders. Informed traders who trade dynamically thus have an incentive not only to trade less aggressively but also to manipulate the market by trading in the wrong direction, undertaking short-term losses to confuse the market and then recouping the losses in the future. Manipulation by an informed trader has been a difficult issue in the literature of market microstructure theory. The contribution to the literature is to prove the uniqueness of equilibrium and characterize it.

Key Words: Market microstructure; Market Price Manipulation; Price Formation; Information asymmetry; Sequential Trade; Bid-Ask Spreads
JEL Classification Numbers: D82, G12.

1 Introduction

This paper studies how stock price manipulation affects the price formation process and information transmission process into a market. In asymmetric information models of financial markets, trading behavior imperfectly reveals the private information held by traders. Back and Baruch (2004) study the equivalence of the two standard reference frameworks in the market microstructure theory: the continuous auction framework, first developed by Kyle (1985) and the sequential trade framework, proposed by Glosten and Milgrom (1985), and show that the equilibrium of the Glosten-Milgrom model is approximately the same as the equilibrium of the Kyle model, when the trade size is small and uninformed trades arrive frequently. They conclude that the continuous-time Kyle model is more tractable than the Glosten-Milgrom model, although most markets are organized as in the sequential trade models. This paper studies the discrete time version of the sequential trade model in Back and Baruch (2004), proves the uniqueness of equilibrium and characterizes it.

This paper considers markets where a risky asset is traded between a market maker who represents a competitive market, strategic informed traders and liquidity traders. There are two types of informed traders. In the beginning of the whole game, nature chooses the liquidation value of the risky asset and tells the informed trader who trade dynamically. With a certain probability the liquidity traders arrive in the market and trade with the market maker for their exogenous liquidity needs. Given the prices that the market maker posts, the dynamic informed trader optimizes his decision over the actions of sell or buy and his optimal strategy chooses the probability distribution over the actions. The market maker updates his belief based on the past trades that he received and posts bid and ask prices, which reflect his expected value of the asset conditional on the sequence of past trades, in equilibrium. Within this framework, we consider a price manipulation defined as trading against information, which means a round-trip trade in equilibrium, because the informed trader once sells/buys against his information and later on buys/sells to recoup the loss.

The main contribution of this paper is to prove the uniqueness of equilibrium and characterize it in the dynamic trading version of the Glosten-Milgrom model. In the original Glosten-Milgrom model, manipulation does not occur because traders can trade only once. In the current paper, the informed trader trade dynamically and in this sense, this paper extends the Glosten-Milgrom model to a dynamic setting. In another aspect, this paper proves the uniqueness of equilibrium which Back and Baruch (2004) assume in their analysis. This paper adds to the literature that brings the canonical model to a dynamics setting and analyzes the equilibrium.

As we can see from the fact that a lot of research has been done by applying the two frameworks, both of the Kyle model and the Glosten-Milgrom model are sufficiently simple and well-behaved that they easily lend themselves to analysis of policy issues and empirical tests.[1] However, neither of them included the possibility of manipulation. In the original Glosten-Milgrom model (see in Glosten and Milgrom (1985)), manipulation does not occur because traders can trade only once. In the Kyle model (see in Kyle (1985)), the informed trader's strategy is monotonic in the sense that he buys the asset when the asset is undervalued given his information and vice versa. Therefore, manipulation is ruled out.

There have been increasing interests in the informed trader's dynamic strategy and market systems. Among others, Brunnermeier and Pedersen (2005) considered dynamic strategic behavior of large traders and show the well-known phenomena, "overshooting"occurs in equilibrium. Back and Baruch (2007) analyzed different market systems by allowing the informed trader's to trade continuously within the Glosten-Milgrom framework. Parlour (1998) presented a one-tick dynamic model of a limit order market and addressed the optimality of different order systems.

A number of authors have considered the definition and possibility of manipulation. The literature started with manipulation by uninformed traders rather than informed traders. Allen and Gale (1992) propose a classification scheme for models of manipulation. They also provide a model of strategic trading in which some equilibria involve manipulation. Using the classification scheme proposed by Allen and Gale (1992), both our model and their model are examples of pure trade-based manipulation, where the informed trader does not announce any information (information-based manipulation) or take any actions (action-based manipulation), except for those that involve trading the asset. Allen and Gorton (1992) also consider a model of pure

[1] See Madhavan (2000) and Biais and Spatt (2005) extensive surveys of the literature.

trade-based uninformed manipulation in which an asymmetry in buys and sells in noise traders trades creates the possibility of manipulation. In every equilibrium of their model, the uninformed manipulator makes zero profits. Jarrow (1992) formulates sufficient conditions for manipulation to be unprofitable. These sufficient conditions are properties of the reduced-form price function.

The first paper that showed manipulation by the informed trader within the discrete-time Glosten-Milgrom framework is Chakraborty and Yilmaz (2004). They show that when the market faces uncertainty about the existence of informed traders in the market and when there are a large number of trading periods, long-lived informed traders will manipulate in every equilibrium. Takayama (2007) provided the lower bound of the number of trading periods for the existence of manipulation in equilibrium. Brunnermeier (2005) considered manipulation in a Kyle-type of market order model. Huberman and Stanzl (2004) study sufficient conditions for the absence of price manipulation. Empirically, Aggarwal and Wu (2006) presented some evidence on stock manipulation in the United States and analyzed price impacts and manipulation in a stock market. Dynamic trading and the effects on price have been an important issue in the literature.

The paper is organized as follows. The second section presents the model. The third section proves the uniqueness of equilibrium. The forth section concludes.

2 The Model

The model in this paper is basically a discrete-time version of Back and Baruch (2004) except that unlike their model, the terminal period is deterministic. Our model is also very similar to Chakraborty and Yilmaz (2004), except here liquidity traders randomly arrive in the stream of informed trading. In this section, we set out a discrete time, sequential trade model of market making. Individuals trade a single risky asset and money with a market maker. Because the market maker is competitive and risk-neutral, these prices are the expected value of the asset conditional on his information at the time of trade.

Trades occur for finitely many periods, denoted by $t = 1, 2, \cdots, T$. Each interval of time accommodates one trade. There is a risky stock and a numeraire in terms of which the stock price is quoted. The terminal value of the risky stock, denoted by $\tilde{v}$, is a random variable, which can take the value 0 or 1. The risk-free interest rate is assumed to be *zero*.

There are two kinds of orders available to traders: sell or buy. Let $A = \{S, B\}$ where S denotes sell order and B denotes buy order. Let $\Delta(A)$ denote the set of probability distributions on A. Let h_t denote the order that the market maker receives in period t, i.e. h_t is the realized order in period t.

There are three classes of risk-neutral market participants: a competitive market maker, an informed trader and a liquidity trader. Trade arises from both informed traders (those who have seen a signal) and uninformed traders. The type of the trader arriving in period t is determined by a random variable $\tilde{\tau}_t$, which takes values from the set $\{i, l\}$. The letters i and l respectively denote the informed type and the liquidity type. The random variables $\{\tilde{\theta}_t : t = 1, ...T\}$ are i.i.d. across the periods $1, ..., T$ and satisfy $\Pr(\tilde{\tau}_t = i) = \mu$. If the trader's type in period t is l, then the demand in that period is determined by the random variable $\tilde{Q}_t$, which takes values from A. The random variables $\{\tilde{Q}_t : t = 1, ..., T\}$ are i.i.d. and satisfy $\Pr(\tilde{Q}_t = a) = \gamma(a) > 0$ for each $a \in A$. For the simplicity of notation, we will write $\gamma(B) = \gamma$ and then $\gamma(S) = 1 - \gamma$. Also, for any given period t, the random variables $\tilde{\tau}_t$, $\tilde{Q}_t$, $\tilde{v}$ are mutually independent.

The private information or type of the trader is determined by a random variable $\tilde{\theta} \in \Theta = \{0, 1\}$. When $\theta = 0$, the trader is informed and knows that the value of the asset is low, $v = 0$. We call this type of trader "low-type" and denote him by L. When $\theta = 1$, the trader is informed and knows that the value of the asset is high, $v = 1$. We call this type of trader "high-type" and denote him by H. Let $N = \{H, L\}$. Only one type of trader is actually chosen by nature to trade for any given play of the game.

Next we describe the details with regard to market maker's pricing strategy and informed traders' trading strategy. To that end, we first need to introduce some notation. First, we set out the space of all possible trading orders. When the traders choose their orders and the market maker posts the bid and ask prices in period t, they know the entire history until and including period $t-1$. A period-t history $h^t := (h_1, ..., h_t)$ is the sequence of realized orders for periods up until $t+1$. The space of all possible period-t histories, $t \geq 1$, is denoted by $H^t := \underbrace{A \times \cdots \times A}_{t \ times}$, and h^t is taken to be the generic element of H^t. For notational convenience, we let $h^0 = \emptyset$.

Knowledge of the game structure and of the parameters of the joint distribution of the traders' state variables is common to all market participants. In each period, market makers post bid and ask prices, equal to the expected value of the asset conditional on the observed history of trades. The trader trades at those prices. Trading happens for finitely many successive periods after which all private information is revealed.

We consider the following game: In the beginning of the whole game, Nature chooses v which is a realization of the risky asset's value. Then, in the beginning of each trading period, with probability μ, an informed trader of type θ will be chosen and with probability $1-\mu$, an informed trader will not be chosen. The timing structure of the trading game is as follows:

1. In period 0, nature chooses the realization $v \in \{0, 1\}$ of the risky asset payoff $\tilde{v}$ and the type of the trader θ. The informed trader observes θ.

2. In successive periods, indexed by $t = 1, ..., T$, having observed the realized trades in periods $1, ..., t-1$, the competitive market maker posts bid and ask prices. Nature chooses a trader (either a dynamic informed trader or a liquidity trader) and the trader learns market maker's price quote.

3. If the trader is informed, he takes the profit-maximizing quote. If the trader is a liquidity trader, he trades according to his liquidity needs. In the end of each trading period, payoff is made to each trader.

4. In period T, the realization of v is publicly disclosed.

A price rule, specifying bid and ask prices that will be posted by the market makers in the beginning of period t, is defined as a function $p_t : \cup_{t=1}^{T-1} H^t \times [\Delta(A)]^2 \to [0,1]^2$ with $p_t = (\beta_t, \alpha_t)$. For each type of the trader, a trading strategy specifies a probability distribution over trades in period $t+1$ with respect to the bid and ask prices p_{t+1} posted in period $t+1$. A strategy for the trader is defined as a function $\sigma_n : \cup_{t=1}^{T-1} H^t \to \Delta(A)$. For each $n \in N = \{H, L\}$ and $a \in A = \{B, S\}$, $\sigma_{na}(h^t)$ be the probability that σ_n assigns to action a after history h^t. That is, $\sigma_{HS}(h^t)$ denotes the probability that the high-type assigns to selling conditional on history h^t.

To determine bid and ask prices to be posted in period t, the market maker updates his prior conditional on the arrival of an order of the relevant type. Let $b : \cup_{t=1}^{T-1} H^t \to \Delta(\{0,1\})$ be the market maker's prior belief at the beginning of period t that the risky asset's value is high conditional on history h^{t-1}. The belief is updated through Bayes' rule; that is, for all $a \in A$,

$$\begin{aligned} b(h^{t-1}, h_t = a) \quad &:= \quad \Pr(\tilde{v} = 1 | h^{t-1}, h_t = a) \\ &= \quad \frac{[\mu\sigma_{Ha}^t(h^{t-1}) + (1-\mu)\gamma] b_t(h^{t-1})}{(1-\mu)\gamma + \mu\sigma_{Ha}^t(h^{t-1}) b_t(h^{t-1}) + \mu\sigma_{La}^t(h^{t-1})(1 - b(h^{t-1}))}. \end{aligned} \tag{1}$$

Definition 1 *A high-type informed trader's strategy is optimal after history h^t if it prescribes a probability distribution $\sigma_H^* \in \Delta(A)$ over $a \in A$ such that*

$$\sigma_H^* \in \arg \max_{\sigma_H \in \Delta(A)} \sum_{s=t}^{T} \left[\sigma_{HB}[1 - \alpha_s(b(h^s))] - \sigma_{HS}[1 - \beta_s(b(h^s))]\right]. \tag{2}$$

Definition 2 *Similarly, a low-type informed trader's strategy is optimal after history h^t if it prescribes a probability distribution $\sigma_L^* \in \Delta(A)$ over $a \in A$ such that*

$$\sigma_L^* \in \arg \max_{\sigma_L \in \Delta(A)} \sum_{s=t}^{T} \left[-\sigma_{LB}\alpha_s(b(h^s)) + \sigma_{HS}\beta_s(b(h^s))\right]. \tag{3}$$

Next we define an equilibrium for our economy:

Definition 3 *An equilibrium consists of a pair of bid and ask prices* $\{p_t^* = (\beta_t^*, \alpha_t^*)\}_{t\in\{1,\cdots,T\}}$*, and informed traders' strategies* $\sigma^* = (\sigma_L^*, \sigma_H^*)$ *such that for all* $t \in \{1, \cdots, T\}$ *and for all* $h^{t-1} \in \cup_{t=1}^{T-1} H^t$*,*

(P1) *the pair of bid and ask prices* p_t^* *satisfies the zero-profit condition with respect to the market maker's posterior belief:* $\alpha_t^*(b(h^{t-1}), \sigma^*) = \mathrm{E}[v|h^{t-1}, h_t = B]$*, and* $\beta_t^*(b(h^{t-1}), \sigma^*) = \mathrm{E}[v|h^{t-1}, h_t = S]$*;*

(P2) *informed traders' strategies* σ_H^* *and* σ_L^* *are optimal given the pair of bid and ask prices* p_t^**;*

(B) *the pair of bid and ask prices* $p_t^* = (\beta_t^*, \alpha_t^*)$ *satisfies Bayes rule (1).*

Therefore, in equilibrium, the following holds: for all $t \in \{1, \cdots, T\}$ and for all $h^{t-1} \in \cup_{t=1}^{T-1} H^t$,

$$\beta_t^*(b(h_{t-1}), \sigma^*) = \frac{[\mu\sigma_{HS}^* + (1-\mu)(1-\gamma)]b(h_{t-1})}{(1-\mu)(1-\gamma) + \mu\sigma_{HS}^* b(h_{t-1}) + \mu\sigma_{LS}^*(1-b(h_{t-1}))}, \tag{4}$$

and

$$\alpha_t^*(b(h_{t-1}), \sigma^*) = \frac{[\mu\sigma_{HB}^* + (1-\mu)\gamma]b(h_{t-1})}{(1-\mu)\gamma + \mu\sigma_{HB}^* b(h_{t-1}) + \mu\sigma_{LB}^*(1-b(h_{t-1}))}. \tag{5}$$

Now, we define a manipulative strategy. We say that a strategy is manipulative if it involves the informed trader undertaking a trade in any period that yields a strictly negative short-term profit. If this occurs in equilibrium, the it means that manipulation enables the informed trader to recoup the short-term losses.

Definition 4 *Given a pair of bid and ask prices* p_t *for some* $t \in \{1, \cdots, T\}$ *and a history* $h_{t-1} \in$ **H***, a strategy* σ_n *is called manipulative in period* t *for the high type if* $\sigma_{HS}(h_{t-1}) > 0$*; or for the low type if* $\sigma_{LB}(h_{t-1}) > 0$*.*

This is the same definition with one in Chakraborty and Yilmaz (2004). Back and Baruch (2004) used the term "bluffing," instead. Basically, we call the situation where the informed trader takes totally mixed strategy, "price manipulation." If totally mixed strategy is taken, the informed trader's strategy assigns strictly positive probability to the order against their information. Now, first we prove the existence of equilibrium.

Theorem 1 *An equilibrium exists.*

Proof: Found in the Appendix. ■

3 The Uniqueness of Equilibrium

By Theorem 1, we know that for each period t, there exists an equilibrium strategy which maximizes the continuation value of the game. In this section, we are going to prove that there exists a unique equilibrium in this model. In order to keep a notation simple, we will eliminate the expression of history h^t and consider a two-period sub-model. Now, let W_H and W_L represent the current value of the game for both traders. Let V_H and V_L represent the continuation value of the remainder of the game for both traders. For example, for arbitrary t, $W_H = V_H^t$ and $V_H = V_H^{t+1}$. Let σ_H and σ_L, both in $\Delta(A)$ denote the mixed strategies for both traders in equilibrium. In this section, for the simplicity of notation, we will write $\alpha = \alpha_t$ and $\beta = \beta_t$.

For the simplicity of notation, let:

$$f_H = \gamma(1-\mu) + \mu\sigma_{HB} \tag{6}$$

and

$$f_L = \gamma(1-\mu) + \mu\sigma_{LB}. \tag{7}$$

Then, f_H denotes the probability that buy order arrives when the state is high and f_L denotes the probability that buy order arrives when the state is low. Note that although we will not write $*$ for all variables, we will study all variables in equilibrium in this section.

In order to prove the uniqueness result, we will provide a sequence of lemmata, propositions and corollaries. Each of the results characterizes the equilibrium. The first step for the uniqueness result is to prove the unique existence of equilibrium supposing the monotonicity and convexity of V_L and V_H in terms of market maker's belief. In other words, we will prove that if monotonic and convex value functions exist in the next period, then in the current period equilibrium strategy profile exists uniquely. Then, we will prove that if the equilibrium strategy exists uniquely in the current period, then W_L and W_H are monotonic and convex, and as a result, we will show that equilibrium exists uniquely for the whole game by mathematical induction.

To begin with, we will consider the equilibrium properties of bid and ask prices and the informed trader's strategy. The following two lemmata characterize those.

Lemma 1 *Fix a history h^t arbitrarily and suppose that $b = b(h^t)$. Suppose that $\sigma = (\sigma_H, \sigma_L)$ is an equilibrium strategy profile. Suppose that V_H is monotonically decreasing in the market maker's prior b and that V_L is monotonically increasing in b. Then, in equilibrium bid-ask spread is strictly positive in $b \in (0,1)$; that is $\alpha(b,\sigma) < \beta(b,\sigma)$.*

Proof:
On the contrary, suppose that for some b, bid-ask spread is negative. That is, $\alpha(b,\sigma) \leq \beta(b,\sigma)$.

Then, we have:

$$
\begin{aligned}
1-\alpha(b,\sigma)+V_H(\alpha(b,\sigma)) &> \beta(b,\sigma)-1+V_H(\beta(b,\sigma)); \quad (8)\\
-\alpha(b)+V_L(\alpha(b,\sigma)) &< \beta(b)+V_L(\beta(b,\sigma)). \quad (9)
\end{aligned}
$$

Suppose that σ_H and σ_L are equilibrium strategies for each type. Then, in equilibrium $\sigma_{HB}=1$ and $\sigma_{LB}=0$. Then, by Bayes rule,

$$
\begin{aligned}
\alpha(b,\sigma) &= \frac{[\mu+(1-\mu)\gamma]b}{(1-\mu)\gamma+\mu b}; \quad (10)\\
\beta(b,\sigma) &= \frac{(1-\mu)(1-\gamma)b}{(1-\mu)(1-\gamma)+\mu(1-b)}. \quad (11)
\end{aligned}
$$

Therefore, we have:

$$\alpha(b,\sigma) > b > \beta(b,\sigma), \quad (12)$$

which contradicts with our assumption. ■

Lemma 2 *Fix a history h^t arbitrarily and suppose that $b=b(h^t)$. Suppose that V_H is monotonically decreasing in the market maker's prior b and that V_L is monotonically increasing in b. Suppose that $\sigma=(\sigma_H,\sigma_L)$ is an equilibrium strategy profile. We have $\alpha(b,\sigma)\geq b$ if and only if $\sigma_{HB}\geq\sigma_{LB}$. Moreover, we have $\beta(b,\sigma)\leq b$ if and only if $\sigma_{HS}\leq\sigma_{LS}$.*

Proof: By Bayes rule, we can obtain the result. ■

Lemma 1 states that in equilibrium, there is no possibility for arbitrage. Lemma 2 states that the high-type buys with a higher probability than the low-type and the low-type sells with a higher probability than the high-type. The following two corollaries are immediate from those two lemmata.

Corollary 1 *Fix a history h^t arbitrarily and suppose that $b=b(h^t)$. Suppose that V_H is monotonically decreasing in the market maker's prior b and that V_L is monotonically increasing in b. In equilibrium, we have: $\alpha(b,\sigma)\geq b\geq\beta(b,\sigma)$, with one of the two inequalities strict.*

Proof: By Lemma 1, the result follows. ■

Corollary 2 *Fix a history h^t arbitrarily and suppose that $b=b(h^t)$. Suppose that V_H is monotonically decreasing in the market maker's prior b and that V_L is monotonically increasing in b. Suppose that $\sigma=(\sigma_H,\sigma_L)$ is an equilibrium strategy profile. In equilibrium, we have: $\sigma_{HB}>\sigma_{LB}$ and $\sigma_{HS}<\sigma_{LS}$.*

Proof: By Corollary 1 and Lemma 2, the result follows. ■

By Corollary 2, we know that in equilibrium, the high-type would not sell with the probability 1 and the low-type would not buy with the probability 1. That means, even if they mix, they would not trade completely in the opposite direction against their information. That leads to the following lemma.

Lemma 3 *Fix a history h^t arbitrarily and suppose that $b = b(h^t)$. Suppose that V_H is monotonically decreasing in the market maker's prior b and that V_L is monotonically increasing in b. Suppose that $\sigma = (\sigma_H, \sigma_L)$ is an equilibrium strategy profile when the belief is b. The followings hold:*

$$\begin{aligned} W_H(b) &= 1 - \alpha(b,\sigma) + V_H(\alpha(b,\sigma)) \\ &\geq \beta(b,\sigma) - 1 + V_H(\beta(b,\sigma)), \end{aligned}$$

and

$$\begin{aligned} W_L(b) &= \beta(b,\sigma) + V_L(\beta(b,\sigma)) \\ &\geq -\alpha(b,\sigma) + V_L(\alpha(b,\sigma)). \end{aligned}$$

Proof: By Corollary 2, we know that in equilibrium, $\sigma_{HB} > 0$ and $\sigma_{LS} > 0$. Therefore, the results follow. ■

Now we turn our attention to the relationship of bid or ask prices between two different beliefs. The following lemma explains this relationship and says that when we compare two equilibrium ask or bid prices corresponding to the two different beliefs, an equilibrium ask or bid price corresponding to a higher belief is higher than the other. For the simplicity of notation, in what follows we will write: $\Gamma_B = (1-\mu)\gamma$ and $\Gamma_S = (1-\mu)(1-\gamma)$.

Lemma 4 *Fix a history h^t arbitrarily and suppose that $b = b(h^t)$. Suppose that $\sigma = (\sigma_H, \sigma_L)$ is an equilibrium strategy profile when the belief is b and $\sigma' = (\sigma'_H, \sigma'_L)$ is an equilibrium strategy profile when the belief is $b - \epsilon$. For every b and sufficiently small ϵ, the followings hold:*

$$\alpha(b,\sigma) - \alpha(b-\epsilon,\sigma') > 0; \tag{13}$$

and

$$\beta(b,\sigma) - \beta(b-\epsilon,\sigma') > 0. \tag{14}$$

Proof:

Since $[0,1]$ is a perfect set, for each point $b \in [0,1]$ we can take a sequence $b^k \to b$ as $k \to \infty$, and also equilibrium strategies associated with each belief, $\sigma_{HB}^k \to \sigma_{HB}$ and $\sigma_{LB}^k \to \sigma_{LB}$ with $\sigma^k = (\sigma_H^k, \sigma_L^k) \in BR(\sigma_H^k, \sigma_L^k)$ and $(\sigma_H, \sigma_L) \in BR(\sigma_H, \sigma_L)$. Then, corresponding to each belief b^k, and the equilibrium strategies (σ_H^k, σ_L^k), by Bayes rule, there is a sequence of ask prices which we denote by $\alpha(b^k)$. Notice that we have: $\alpha(b^k, \sigma^k) \to \alpha(b, \sigma)$ as $k \to \infty$. Then, we have:

$$
\begin{aligned}
&\frac{\alpha(b^k,\sigma^k) - \alpha(b,\sigma)}{b^k - b} \\
&= \frac{\Gamma_B^2(b^k - b) + \mu\Gamma_B(b^k(1-b)(\sigma_{HB}^k + \sigma_{LB}) - b(1-b^k)(\sigma_{HB} + \sigma_{LB}^k))}{(b^k - b)[\Gamma_B + \mu b\sigma_{HB} + \mu(1-b)\sigma_{LB}][\Gamma_B + \mu b^k\sigma_{HB}^k + \mu(1-b^k)\sigma_{LB}^k]} \\
&\qquad + \frac{\mu^2(b^k(1-b)\sigma_{HB}^k\sigma_{LB} - b(1-b^k)\sigma_{HB}\sigma_{LB}^k)}{(b^k - b)[\Gamma_B + \mu b\sigma_{HB} + \mu(1-b)\sigma_{LB}][\Gamma_B + \mu b^k\sigma_{HB}^k + \mu(1-b^k)\sigma_{LB}^k]}.
\end{aligned}
$$

Thus, we obtain:

$$
\begin{aligned}
\lim_{b^k \to b} \frac{\alpha(b^k,\sigma^k) - \alpha(b,\sigma)}{b^k - b} &= \frac{\Gamma_B^2 + \mu\Gamma_B(\sigma_{HB} + \sigma_{LB}) + \mu^2\sigma_{HB}\sigma_{LB}}{[\Gamma_B + \mu b\sigma_{HB} + \mu(1-b)\sigma_{LB}]^2} \lim_{b^k \to b} \frac{b^k - b}{b^k - b} \\
&= \frac{\Gamma_B^2 + \mu\Gamma_B(\sigma_{HB} + \sigma_{LB}) + \mu^2\sigma_{HB}\sigma_{LB}}{[\Gamma_B + \mu b\sigma_{HB} + \mu(1-b)\sigma_{LB}]^2}.
\end{aligned}
$$

Similarly, for a bid-price,

$$
\begin{aligned}
&\frac{\beta(b^k,\sigma^k) - \beta(b,\sigma)}{b^k - b} \\
&= \frac{\Gamma_S^2(b^k - b) + \mu\Gamma_S(b^k(1-b)(\sigma_{LS}^k + \sigma_{HS}) - b(1-b^k)(\sigma_{LS} + \sigma_{HS}^k)) + \mu^2(b^k(1-b)\sigma_{LS}^k\sigma_{HS} - b(1-b^k)\sigma_{LS}\sigma_{HS}^k)}{(b^k - b)[\Gamma_S + \mu(1-b)\sigma_{LS} + \mu b\sigma_{HS}][\Gamma_S + \mu(1-b^k)\sigma_{LS}^k + \mu b^k\sigma_{HS}^k]}.
\end{aligned}
$$

Therefore, we obtain:

$$
\lim_{b^k \to b} \frac{\beta(b^k,\sigma^k) - \beta(b,\sigma)}{b^k - b} = \frac{\Gamma_S^2 + \mu\Gamma_S(\sigma_{LS} + \sigma_{HS}) + \mu^2\sigma_{LS}\sigma_{HS}}{[\Gamma_S + \mu b\sigma_{HS} + \mu(1-b)\sigma_{LS}]^2}.
$$

Thus, we conclude that: $\lim_{b^k \to b} \frac{\alpha(b^k,\sigma^k) - \alpha(b,\sigma)}{b^k - b}$ and $\lim_{b^k \to b} \frac{\beta(b^k,\sigma^k) - \beta(b,\sigma)}{b^k - b}$ are greater than *zero* and the result follows. ■

Next, we consider the property of the value functions. Manipulation occurs in order to affect the future prices. This effect on price must be related to the future value functions, and otherwise there is no point of taking manipulative strategy. The next two lemmata give a sufficient condition for manipulation to occur.

Lemma 5 *Fix a history h^t arbitrarily and suppose that $b = b(h^t)$. Suppose that V_H is monotonically decreasing in the market maker's prior b and that V_L is monotonically increasing in b.*

Suppose that $\sigma = (\sigma_H, \sigma_L)$ is an equilibrium strategy profile when the belief is b and $\sigma' = (\sigma'_H, \sigma'_L)$ is an equilibrium strategy profile when the belief is $b - \epsilon$. If the low-type takes a manipulative strategy at b, for a sufficiently small ϵ the following holds:

$$\frac{V_L(\alpha(b,\sigma)) - V_L(\alpha(b-\epsilon,\sigma'))}{\alpha(b,\sigma) - \alpha(b-\epsilon,\sigma')} > 1. \tag{15}$$

On the other hand, suppose that $\sigma = (\sigma_H, \sigma_L)$ is an equilibrium strategy profile when the belief is b and $\sigma'' = (\sigma''_H, \sigma''_L)$ is an equilibrium strategy profile when the belief is $b + \epsilon$. If the high-type takes a manipulative strategy at b, for a sufficiently small ϵ the following holds:

$$\frac{V_H(\beta(b,\sigma)) - V_H(\beta(b+\epsilon,\sigma''))}{\beta(b,\sigma) - \beta(b+\epsilon,\sigma'')} < -1. \tag{16}$$

Proof for the Low-type:

On the contrary, suppose that for ϵ sufficiently small, the following holds:

$$\frac{V_L(\alpha(b,\sigma)) - V_L(\alpha(b-\epsilon,\sigma'))}{\alpha(b,\sigma) - \alpha(b-\epsilon,\sigma')} \leq 1. \tag{17}$$

By assumption, we have:

$$-\alpha(b,\sigma) + V_L(\alpha(b,\sigma)) = \beta(b,\sigma) + V_L(\beta(b,\sigma)). \tag{18}$$

Then, at $b - \epsilon$, by Lemma 3 we have:

$$-\alpha(b-\epsilon,\sigma') + V_L(\alpha(b-\epsilon,\sigma')) \leq \beta(b-\epsilon,\sigma') + V_L(\beta(b-\epsilon,\sigma')). \tag{19}$$

This means:

$$\begin{aligned}&-\alpha(b-\epsilon,\sigma') + \alpha(b,\sigma) + V_L(\alpha(b-\epsilon,\sigma')) - V_L(\alpha(b,\sigma)) \\ &\qquad \leq \beta(b-\epsilon,\sigma') - \beta(b,\sigma) + V_L(\beta(b-\epsilon,\sigma')) - V_L(\beta(b,\sigma)).\end{aligned} \tag{20}$$

Then, by our assumption, we must have:

$$-\alpha(b-\epsilon,\sigma') + \alpha(b,\sigma) + V_L(\alpha(b-\epsilon,\sigma')) - V_L(\alpha(b,\sigma)) \geq 0. \tag{21}$$

Since V_L are monotonically increasing and by Lemma 4, we must have:

$$0 > \beta(b-\epsilon,\sigma') - \beta(b,\sigma) + V_L(\beta(b-\epsilon,\sigma')) - V_L(\beta(b,\sigma)). \tag{22}$$

Thus, (20) is impossible. □

Proof for the High-Type:

On the contrary, suppose that for ϵ sufficiently small, the following holds:

$$\frac{V_H(\beta(b,\sigma)) - V_H(\beta(b+\epsilon,\sigma''))}{\beta(b,\sigma) - \beta(b+\epsilon,\sigma'')} \geq -1. \tag{23}$$

By assumption, we have:

$$1 - \alpha(b,\sigma) + V_H(\alpha(b,\sigma)) = \beta(b,\sigma) - 1 + V_H(\beta(b,\sigma)). \tag{24}$$

Then at $b+\epsilon$, by Lemma 3 we have:

$$1 - \alpha(b+\epsilon,\sigma'') + V_H(\alpha(b+\epsilon,\sigma'')) \geq \beta(b+\epsilon,\sigma'') - 1 + V_H(\beta(b+\epsilon,\sigma'')). \tag{25}$$

This means:

$$\begin{aligned} &-\alpha(b+\epsilon,\sigma'') + \alpha(b,\sigma) + V_H(\alpha(b+\epsilon,\sigma'')) - V_H(\alpha(b,\sigma)) \\ &\qquad \geq \beta(b+\epsilon,\sigma'') - \beta(b,\sigma) + V_H(\beta(b+\epsilon,\sigma'')) - V_H(\beta(b,\sigma)). \end{aligned} \tag{26}$$

Then, by our assumption, we must have:

$$\beta(b+\epsilon,\sigma'') - \beta(b,\sigma) + V_H(\beta(b+\epsilon,\sigma'')) - V_H(\beta(b,\sigma)) \geq 0. \tag{27}$$

Since V_H is monotonically decreasing and by Lemma 4, we must have:

$$0 > -\alpha(b+\epsilon,\sigma'') + \alpha(b,\sigma) + V_H(\alpha(b+\epsilon,\sigma)) - V_H(\alpha(b,\sigma)). \tag{28}$$

Thus, (26) is impossible. $\square$ $\blacksquare$

Lemma 6 *Fix a history h^t arbitrarily and suppose that $b = b(h^t)$. Suppose that V_H is monotonically decreasing in the market maker's prior b and that V_L is monotonically increasing in b. Suppose that V_H and V_L are strictly convex. Suppose that σ is an equilibrium strategy profile when the belief is b. Then the followings hold:*

$$V_L'(\alpha(b,\sigma)) > 1; \tag{29}$$

and

$$V_H'(\beta(b,\sigma)) < -1. \tag{30}$$

Proof:

By Lemma 5, (15) and (16) hold for any ϵ. Therefore, since V_H and V_L are strictly convex, we obtain the results. $\blacksquare$

A simple intuition of Lemma 6 is that if manipulation occurs, the effect for the future payoff has to be large enough. In an extreme case, if the value function is completely flat, then even if the informed trader suffers the short-term loss, the future payoff would not change. Therefore, if the value function is completely flat, manipulation would not occur. In other words, if manipulation occurs, the value function must be steep enough. Lemma 6 gave a critical value for manipulation.

Now that we have proved all the necessary results to prove the uniqueness of equilibrium, we consider possible cases of manipulation by the informed traders in equilibrium so that we can grasp more ideas about how equilibrium works. In equilibrium, there are four cases; that is, only the high-type totally mixes, only the low-type totally mixes, both totally mix, and neither totally mixes. Lemma 7 will prove the uniqueness of equilibrium strategy for the first or second case, in which only one of them totally mixes. Lemma 8 and Lemma 9 will show the uniqueness of equilibrium for the third case.

Lemma 7 *Fix a history h^t arbitrarily and suppose that $b = b(h^t)$. Suppose that V_H is monotonically decreasing in the market maker's prior b and that V_L is monotonically increasing in b. Suppose that V_H and V_L are strictly convex. Suppose that in equilibrium one type totally mixes. Then, the equilibrium strategy exists uniquely.*

Proof:
Since the argument is symmetric, we will prove the result for only the high-type. Suppose that σ_H is totally mixed strategies. Then, the high-type must be indifferent between purchase and sell. Therefore, the following hold:

$$1 - \alpha(b,\sigma) + V_H(\alpha(b,\sigma)) = \beta(b) - 1 + V_H(\beta(b,\sigma)). \quad (31)$$

We will show that given b and σ_L there is a unique pair of strategies σ_H satisfying the above (31).

On the contrary, suppose that there are different strategies $\hat{\sigma}_H$ satisfying (31) with the prices $\alpha(b,\hat{\sigma})$ and $\beta(b,\hat{\sigma})$. Now, suppose that: $\sigma_{HB} < \hat{\sigma}_{HB}$. Then, we have: $\alpha(b,\hat{\sigma}) > \alpha(b,\sigma)$ and $\beta(b,\hat{\sigma}) < \beta(b,\sigma)$.

Then, we have:

$$1 - \alpha(b,\hat{\sigma}) + V_H(\alpha(b,\hat{\sigma})) \geq \beta(b,\hat{\sigma}) - 1 + V_H(\beta(b,\hat{\sigma})). \quad (32)$$

Thus, we have:

$$\alpha(b,\hat{\sigma}) - \alpha(b,\sigma) - V_H(\alpha(b,\hat{\sigma})) + V_H(\alpha(b,\sigma)) \leq \beta(b,\sigma) - \beta(b,\hat{\sigma}) + V_H(\beta(b,\sigma)) - V_H(\beta(b,\hat{\sigma})). \quad (33)$$

By arranging the above, we have:

$$\alpha(b,\hat{\sigma})-\alpha(b,\sigma)+[V_H(\alpha(b,\sigma))-V_H(\alpha(b,\hat{\sigma}))]+[V_H(\beta(b,\hat{\sigma}))-V_H(\beta(b,\sigma))] \leq \beta(b,\sigma)-\beta(b,\hat{\sigma}). \quad (34)$$

However, since the slope of the Value for the high-type must be greater than 1 at bid price, we have:

$$[V_H(\beta(b,\hat{\sigma})) - V_H(\beta(b,\sigma))] > \beta(b,\sigma) - \beta(b,\hat{\sigma}). \quad (35)$$

Since V_H is strictly decreasing, it is impossible for (34) to hold. ■

Lemma 8 *Fix a history h^t arbitrarily and suppose that $b = b(h^t)$. Suppose that V_H is monotonically decreasing in the market maker's prior b and that V_L is monotonically increasing in b. Suppose that V_H and V_L are strictly convex. Suppose that in equilibrium, both types totally mix. Then, the equilibrium bid and ask prices are unique.*

Proof:
If both types mix at the same time, the following holds:

$$1 - \alpha(b,\sigma) + V_H(\alpha(b,\sigma)) = \beta(b,\sigma) - 1 + V_H(\beta(b,\sigma)), \quad (36)$$

and

$$-\alpha(b,\sigma) + V_L(\alpha(b,\sigma)) = \beta(b,\sigma) + V_L(\beta(b,\sigma)). \quad (37)$$

Consequently, the following must be true:

$$[V_L(\alpha(b,\sigma)) - V_H(\alpha(b,\sigma))] - [V_L(\beta(b,\sigma)) - V_H(\beta(b,\sigma))] = 2. \quad (38)$$

Consider the difference between the bid and ask prices (that is, bid-ask spread). Notice that $\frac{f_L}{f_H} \in [\frac{\gamma(1-\mu)}{\gamma(1-\mu)+\mu}, 1]$ and $\frac{1-f_L}{1-f_H} \in [1, \frac{(1-\gamma)(1-\mu)+\mu}{(1-\gamma)(1-\mu)}]$. By Bayes rule, we have:

$$\alpha(b,\sigma) = \frac{f_H b}{f_H b + (1-b) f_L} = \frac{b}{b + (1-b)\frac{f_L}{f_H}}, \quad (39)$$

and

$$\beta(b,\sigma) = \frac{(1-f_H)b}{(1-f_H)b + (1-b)(1-f_L)} = \frac{b}{b + (1-b)\frac{1-f_L}{1-f_H}}. \quad (40)$$

Therefore, when $\frac{f_L}{f_H} = 1$ and $\frac{1-f_L}{1-f_H} = 1$ corresponding to the strategies $\sigma_{LB} = \sigma_{HB}$, the bid-ask spread is the smallest and then we have:

$$\underline{S}(b,\sigma) \equiv \alpha(b,\sigma) - \beta(b,\sigma) = b - b = 0. \quad (41)$$

On the other hand, when $\frac{f_L}{f_H} = \frac{\gamma(1-\mu)}{\gamma(1-\mu)+\mu}$ and $\frac{1-f_L}{1-f_H} = \frac{(1-\gamma)(1-\mu)+\mu}{(1-\gamma)(1-\mu)}$ corresponding to the strategies $\sigma_{LB} = 0$ and $\sigma_{HB} = 1$, the bid-ask spread is the largest and then we have:

$$\bar{S}(b,\sigma) \equiv \alpha(b,\sigma) - \beta(b,\sigma) = \frac{b}{b + (1-b)\frac{\gamma(1-\mu)}{\gamma(1-\mu)+\mu}} - \frac{b}{b + (1-b)\frac{(1-\gamma)(1-\mu)+\mu}{(1-\gamma)(1-\mu)}}. \tag{42}$$

Since the equilibrium bid-ask spread $S(b,\sigma)$ must be between $\underline{S}(b,\sigma)$ and $\bar{S}(b,\sigma)$, if both types mix, $\bar{S}(b,\sigma)$ must be strictly greater than 2. Otherwise, (38) would not hold for any $\alpha(b,\sigma)$ and $\beta(b,\sigma)$. On the other hand, if $\bar{S}(b,\sigma)$ is strictly greater than 2, then by the Intermediate Value Function Theorem there will be a pair of bid and ask prices $\alpha(b,\sigma)$ and $\beta(b,\sigma)$ which satisfies (38). Now, we will prove that there is only one pair of bid and ask prices $\alpha(b,\sigma)$ and $\beta(b,\sigma)$ which satisfies (38).

We define: for $\alpha \in [0,1]$ and $\beta \in [0,1]$,

$$H(\alpha,\beta) = V_H(\alpha) - V_H(\beta) + 2 - \alpha - \beta, \tag{43}$$

and

$$L(\alpha,\beta) = V_L(\alpha) - V_L(\beta) - \alpha - \beta. \tag{44}$$

Also, we define:

$$J(\alpha,\beta) \equiv \begin{pmatrix} H(\alpha,\beta) \\ L(\alpha,\beta) \end{pmatrix}. \tag{45}$$

By Lemma 6 we know that if the high-type mixes, then

$$V_H'(\beta) < -1, \tag{46}$$

and if the low-type mixes, then

$$V_L'(\alpha) > 1. \tag{47}$$

Now, we consider the determinant of the following matrix:

$$\mathrm{d}J(\alpha,\beta) \equiv \begin{pmatrix} \frac{\partial H}{\partial \alpha} & \frac{\partial H}{\partial \beta} \\ \frac{\partial L}{\partial \alpha} & \frac{\partial L}{\partial \beta} \end{pmatrix}. \tag{48}$$

Then, since V_H is decreasing, V_L is increasing and by Lemma 5, we obtain:

$$\begin{aligned} \frac{\partial H}{\partial \alpha} &= V_H'(\alpha) - 1 < 0; \\ \frac{\partial H}{\partial \beta} &= -V_H'(\beta) - 1 > 0; \\ \frac{\partial L}{\partial \alpha} &= V_L'(\alpha) - 1 > 0; \\ \frac{\partial L}{\partial \beta} &= -V_L'(\beta) - 1 < 0;. \end{aligned}$$

Therefore,

$$\begin{aligned}
\det(\mathrm{d}J) &= \frac{\partial H}{\partial \alpha} \times \frac{\partial L}{\partial \beta} - \frac{\partial H}{\partial \beta} \times \frac{\partial L}{\partial \alpha} && (49)\\
&= -[V_H'(\alpha) - 1] \times [V_L'(\beta) + 1] + [V_H'(\beta) + 1] \times [V_L'(\alpha) - 1]. && (50)
\end{aligned}$$

Thus, $\det(\mathrm{d}J) > 0$ if and only if:

$$[V_H'(\beta) + 1] \times [V_L'(\alpha) - 1] > [V_H'(\alpha) - 1] \times [V_L'(\beta) + 1]. \tag{51}$$

Then, (51) holds if and only if:

$$\frac{[V_H'(\beta) + 1]}{[V_H'(\alpha) - 1]} < \frac{[V_L'(\beta) + 1]}{[V_L'(\alpha) - 1]}. \tag{52}$$

Then, (52) holds if and only if:

$$\frac{|V_H'(\beta) + 1|}{|V_H'(\alpha) - 1|} < \frac{[V_L'(\beta) + 1]}{[V_L'(\alpha) - 1]}. \tag{53}$$

Notice that since V_H is a decreasing function and V_L is a increasing function, we have:

$$|V_H'(\beta) + 1| < [V_L'(\beta) + 1], \tag{54}$$

and

$$|V_H'(\alpha) - 1| > [V_L'(\alpha) - 1]. \tag{55}$$

Therefore, (53) holds and as a result, we can conclude that $\mathrm{d}J$ has a strictly positive determinant. Since the elements in the upper left corner of $\mathrm{d}J$ and the lower right corner of $\mathrm{d}J$ are both strictly negative, we conclude that $\mathrm{d}J$ is negative definite. Take two distinct $p_1 = (\alpha_1, \beta_1)$ and $p_2 = (\alpha_2, \beta_2)$. Then, we have:

$$\begin{aligned}
< p_1 - p_2, J(p_1) - J(p_2) > &= < p_1 - p_2, \int_0^1 \mathrm{d}J(p_2 + t(p_1 - p_2))(p_1 - p_2)\mathrm{d}t > \\
&= \int_0^1 (p_1 - p_2)^T \mathrm{d}J(p_2 + t(p_1 - p_2))(p_1 - p_2)\mathrm{d}t \\
&< 0.
\end{aligned}$$

Therefore, we have: $J(p_1) \neq J(p_2)$, which means:

$$\begin{pmatrix} H(\alpha_1, \beta_1) \\ L(\alpha_1, \beta_1) \end{pmatrix} \neq \begin{pmatrix} H(\alpha_2, \beta_2) \\ L(\alpha_2, \beta_2) \end{pmatrix}. \tag{56}$$

Therefore, there exists only one pair of α and β which satisfies: $H(\alpha, \beta) = 0$ and $L(\alpha, \beta) = 0$. Finally, we conclude that there is only one pair of α and β which satisfies (36) and (37). This completes our proof. ■

Lemma 9 *If the equilibrium bid and ask prices are unique, the equilibrium strategies are unique.*

Proof:

Suppose that in equilibrium, there are two different pairs of strategies, σ and $\hat{\sigma}$. Now on the contrary to our conclusion of this lemma, suppose that $\alpha(b, \hat{\sigma}) = \alpha(b, \sigma)$ and $\beta(b, \hat{\sigma}) = \beta_t(b, \sigma)$.

By Bayes rule, we can write:

$$\alpha(b, \sigma) = \frac{f_H b}{f_H b + (1-b) f_L}. \tag{57}$$

Similarly with f_H and f_L, we define $\hat{f}_H$ and $\hat{f}_L$ associated with $\hat{\sigma}_{LB}$ and $\hat{\sigma}_{HB}$. Then the following holds:

$$\alpha(b, \hat{\sigma}) = \frac{\hat{f}_H b}{\hat{f}_H b + (1-b) \hat{f}_L}. \tag{58}$$

By equating (57) and (58) we must have:

$$\hat{f}_H f_L = \hat{f}_L f_H. \tag{59}$$

Similarly for the bid-price, we have:

$$\beta(b, \sigma) = \frac{(1-f_H)b}{(1-f_H)b + (1-b)(1-f_L)}, \tag{60}$$

and

$$\beta(b, \hat{\sigma}) = \frac{(1-\hat{f}_H)b}{(1-\hat{f}_H)b + (1-b)(1-\hat{f}_L)}. \tag{61}$$

By equating (60) and (61) we must have:

$$(1-\hat{f}_H)(1-f_L) = (1-\hat{f}_L)(1-f_H). \tag{62}$$

Combining the equations (59) and (62) gives

$$\hat{f}_H - f_H = f_L - \hat{f}_L. \tag{63}$$

Let the difference in (63) by Δ. Then, by substituting it into (59) we obtain:

$$(f_H + \Delta) f_L = (f_L + \Delta) f_H. \tag{64}$$

Therefore, we must have $f_H = f_L$ and $\hat{f}_H = \hat{f}_L$. Conversely, if $f_H = f_L$ and $\hat{f}_H = \hat{f}_L$, then $\beta(b, \hat{\sigma}) = \beta(b, \sigma) = b$ and $\alpha(b, \hat{\sigma}) = \alpha(b, \sigma) = b$. This contradicts with Lemma 1. ■

Lemma 10 *Fix a history h^t arbitrarily and suppose that $b = b(h^t)$. Suppose that V_H is monotonically decreasing in the market maker's prior b and that V_L is monotonically increasing in b. Suppose that V_H and V_L are strictly convex. The equilibrium exists uniquely.*

Proof:
Four cases can arise in equilibrium; that is, only the high-type totally mixes, only the low-type totally mixes, both manipulate, and neither totally mixes. In the first or second case, in which one of them totally mixes, by Lemma 7, the equilibrium strategy is uniquely determined. Therefore, the corresponding price is uniquely determined by Bayes rule. In the third case, by Lemma 8 and Lemma 9, the equilibrium exists uniquely. If both do not totally mix, the equilibrium strategy is $\sigma_{HB} = 1$ and $\sigma_{LB} = 0$ and thus the corresponding price is uniquely determined by Bayes rule. In the end, we conclude that equilibrium exists uniquely in either case. ■

So far, we have proved that if V_H and V_L are monotonic, and V_H and V_L are strictly convex, then in the current period the equilibrium exists uniquely. In order to complete our proof for the whole game, we have to prove that actually W_H and W_L are monotonic, and W_H and W_L are strictly convex. In order to do so, we will start with the monotonicity of W_H and W_L.

Lemma 11 *Fix a history h^t arbitrarily and suppose that $b = b(h^t)$. Suppose that V_H is monotonically decreasing in the market maker's prior b and that V_L is monotonically increasing in b. Suppose that V_H and V_L are strictly convex. Then, equilibrium exists uniquely and the high-type's value function W_H is monotonically decreasing and the low type's value function W_L is monotonically increasing in terms of the market maker's prior belief b.*

Proof:
· **When** $t = T$
Since this is the last chance to trade, both types trade on their information. Therefore,

$$\alpha(b,\sigma) = \frac{[\mu + \Gamma_B]b}{\Gamma_B + \mu b}.$$

Thus,

$$W_H(b) = 1 - \frac{[\mu + \Gamma_B]b}{\Gamma_B + \mu b} = \frac{(1-b)\Gamma_B}{\Gamma_B + \mu b}. \tag{65}$$

Therefore, we conclude that W_H is strictly decreasing in b. □

· **When** $t = 1, \cdots, T-1$
By Lemma 10, equilibrium exists uniquely. Let $b > b'$, and σ' denotes the equilibrium strategy when the belief is b'. Then, we have:

$$\begin{aligned} W_H(b) &= 1 - \alpha(b,\sigma) + V_H(\alpha(b,\sigma)) \\ &< 1 - \alpha(b',\sigma') + V_H(\alpha(b',\sigma')) \\ &= W_H(b'). \end{aligned}$$

This completes our proof. ■

Now we will prove the strict convexity of W_H and W_L. In order to do so, first we will consider the shapes of bid and ask prices.

Proposition 1 *Fix a history h^t arbitrarily and suppose that $b = b(h^t)$. Suppose that V_H is monotonically decreasing in the market maker's prior b and that V_L is monotonically increasing in b. Suppose that V_H and V_L are strictly convex. Then, ask price is strictly concave and bid price is strictly convex in terms of b.*

Proof:
By Lemma 10, equilibrium exists uniquely. By Lemma 4, we have:

$$\alpha'(b,\sigma) = \lim_{b^k \to b} \frac{\alpha(b^k) - \alpha(b)}{b^k - b} = \frac{\Gamma_B^2 + \mu\Gamma_B(\sigma_{HB} + \sigma_{LB}) + \mu^2\sigma_{HB}\sigma_{LB}}{[f_H b + f_L(1-b)]^2},$$

and

$$\beta'(b,\sigma) = \lim_{b^k \to b} \frac{\beta(b^k) - \beta(b)}{b^k - b} = \frac{\Gamma_S^2 + \mu\Gamma_S(\sigma_{LS} + \sigma_{HS}) + \mu^2\sigma_{LS}\sigma_{HS}}{[(1-f_H)b + (1-f_L)(1-b)]^2}.$$

Similarly with the proof of Proposition 4, we can take a sequence $b^k \to b$ as $k \to \infty$, and also the equilibrium strategies associated with each belief, $\sigma_{HB}^k \to \sigma_{HB}$ and $\sigma_{LB}^k \to \sigma_{LB}$ with $\sigma^k = (\sigma_H^k, \sigma_L^k) \in BR(\sigma_H^k, \sigma_L^k)$ and $(\sigma_H, \sigma_L) \in BR(\sigma_H, \sigma_L)$.

We also denote f_H^k and f_L^k, and f_H and f_L defined in (6) and (7) associated with σ^k and σ. Then we have $f_H^k \to f_H$ and $f_L^k \to f_L$ as $k \to \infty$. Now we consider:

$$\begin{aligned}
\alpha''(b,\sigma) &\equiv \lim_{b^k \to b} \frac{\alpha'(b^k,\sigma^k) - \alpha'(b,\sigma)}{b^k - b} \\
&= \lim_{b^k \to b} \frac{[\Gamma_B^2 + \mu\Gamma_B(\sigma_{HB}^k + \sigma_{LB}^k) + \mu^2\sigma_{HB}^k\sigma_{LB}^k][f_H b + f_L(1-b)]^2}{[f_H b + f_L(1-b)]^2 \cdot [f_H^k b^k + f_L^k(1-b^k)]^2} \\
&\qquad - \lim_{b^k \to b} \frac{[\Gamma_B^2 + \mu\Gamma(\sigma_{HB} + \sigma_{LB}) + \mu^2\sigma_{HB}\sigma_{LB}][f_H^k b^k + f_L^k(1-b^k)]^2}{[f_H b + f_L(1-b)]^2 \cdot [f_H^k b^k + f_L^k(1-b^k)]^2} \\
&= \frac{[\Gamma_B^2 + \mu\Gamma_B(\sigma_{HB} + \sigma_{LB}) + \mu^2\sigma_{HB}\sigma_{LB}]}{[f_H b + f_L(1-b)]^4} \\
&\qquad \times \lim_{b^k \to b} \frac{[f_H b + f_L(1-b)] + [f_H^k b^k + f_L^k(1-b^k)]][[f_H b + f_L(1-b)] - [f_H^k b^k + f_L^k(1-b^k)]]}{b^k - b} \\
&= 2\frac{[\Gamma_B^2 + \mu\Gamma_B(\sigma_{HB} + \sigma_{LB}) + \mu^2\sigma_{HB}\sigma_{LB}]}{[f_H b + f_L(1-b)]^3} \lim_{b^k \to b} \frac{[f_H b + f_L(1-b)] - [f_H^k b^k + f_L^k(1-b^k)]}{b^k - b} \\
&= 2\frac{[\Gamma_B^2 + \mu\Gamma_B(\sigma_{HB}(b) + \sigma_{LB}) + \mu^2\sigma_{HB}\sigma_{LB}]}{[f_H b + f_L(1-b)]^3} \lim_{b^k \to b} \frac{\mu[\sigma_{HB} b - \sigma_{HB}^k b^k] + \mu[\sigma_{LB}(1-b) - \sigma_{LB}^k(1-b^k)]}{b^k - b} \\
&= 2\mu \frac{[\Gamma_B^2 + \mu\Gamma_B(\sigma_{HB} + \sigma_{LB}) + \mu^2\sigma_{HB}\sigma_{LB}]}{[f_H b + f_L(1-b)]^3} \cdot [\sigma_{LB} - \sigma_{HB}] < 0.
\end{aligned}$$

Similarly, for a bid-price,

$$\beta''(b,\sigma) \equiv \lim_{b^k \to b} \frac{\beta'(b^k,\sigma^k) - \beta'(b,\sigma)}{b^k - b}$$
$$= 2\mu \frac{[\Gamma_S^2 + \mu\Gamma_S(\sigma_{HS} + \sigma_{LS}) + \mu^2\sigma_{HS}\sigma_{LS}]}{[(1-f_H)b + (1-f_L)(1-b)]^3} \cdot [\sigma_{LS} - \sigma_{HS}] > 0.$$

Therefore, we obtain the desired results. ■

Theorem 2 *Fix a history h^t arbitrarily and suppose that $b = b(h^t)$. Suppose that V_H is monotonically decreasing in the market maker's prior b and that V_L is monotonically increasing in b. Suppose that V_H and V_L are strictly convex. The high-type's value W_H and the low type's value W_L are strictly convex in terms of the market maker's prior belief b.*

Proof:

· **When** $t = T$

By taking the second derivative of (66), we can conclude that:

$$\frac{\mathrm{d}^2 W_H}{\mathrm{d}b^2} > 0.$$

Since W_H is strictly decreasing, we can conclude that W_H is convex in b. □

· **When** $t = 1, \cdots, T-1$

Suppose that V_H is strictly convex in b. By Lemma 10, equilibrium exists uniquely. Suppose that for $r, b_1, b_2 \in [0,1]$, σ_1, σ_2 and $\bar{\sigma}$ is respectively the equilibrium strategy when the belief is b_1, b_2 and $rb_1 + (1-r)b_2$. Then, we have:

$$\begin{aligned}
W_H(rb_1 + (1-r)b_2) &= 1 - \alpha(rb_1 + (1-r)b_2, \bar{\sigma}) + V_H(\alpha(rb_1 + (1-r)b_2, \bar{\sigma})) \\
&< 1 - r\alpha(b_1,\sigma_1) - (1-r)\alpha(b_2,\sigma_2) + V_H(r\alpha(b_1,\sigma_1) + (1-r)\alpha(b_2,\sigma_2)) \\
&\quad (\because \alpha \text{ is strictly concave and } V_H \text{ is strictly decreasing.}) \\
&< 1 - r\alpha(b_1,\sigma_1) - (1-r)\alpha(b_2,\sigma_2) + rV_H(\alpha(b_1,\sigma_1)) + (1-r)V_H(\alpha(b_2,\sigma_2)) \\
&\quad (\because V_H \text{ is strictly convex.}) \\
&= r\left[1 - \alpha(b_1,\sigma_1) + V_H(\alpha(b_1,\sigma_1))\right] + (1-r)\left[1 - \alpha(b_2,\sigma_2) + V_H(\alpha(b_2,\sigma_2))\right] \\
&= rW_H(b_1) + (1-r)W_H(b_2).
\end{aligned}$$

Thus, we conclude that $W_H(b)$ is strictly convex. Symmetrically, we can also prove that $W_L(b)$ is strictly convex. This completes our proof. □ ■

Theorem 3 *The equilibrium exists uniquely.*

Proof:
We will prove this inductively. Consider the second last period $t = T$. Then, both informed traders trade on information. Therefore,

$$\alpha_T(b, \sigma) \quad = \quad \frac{[\mu + \Gamma_B]b}{\Gamma_B + \mu b}.$$

Thus,

$$V_H^T(b) = 1 - \frac{[\mu + \Gamma_B]b}{\Gamma_B + \mu b} = \frac{(1-b)\Gamma_B}{\Gamma_B + \mu b}. \tag{66}$$

Thus, V_H^T is strictly decreasing in b. Moreover, V_H^T is strictly convex in b. Thus, the equilibrium strategy exists uniquely in period $t = T - 1$ by Lemma 10. Thus, there exists a unique V_H^{T-1} which is monotonically decreasing and strictly convex in the market maker's belief at period $T-1$ by Lemma 11. Thus, the equilibrium strategy exists uniquely in period $t = T - 2$ by Lemma 10. Inductively, we can obtain the desired result. ■

In this section, we proved the uniqueness of equilibrium. At the same time, we also proved some interesting properties of the value functions, and bid and ask prices. The high-type's value function W_H is monotonically decreasing and the low type's value function W_L is monotonically increasing in terms of the market maker's prior belief b by Lemma 11 and Theorem 3. The high-type's value W_H the low type's value W_L is strictly convex in terms of the market maker's prior belief b by Lemma 11 and Theorem 3. Moreover, we proved that equilibrium bid and ask prices are monotonically increasing, bid price is strictly convex and ask price is strictly concave. Although there is a difference about discrete or continuous time, and the deterministic or stochastic terminal period, those properties are one that Back and Baruch (2004) showed in a numerical experiments. In this section, we proved those properties.

4 Concluding Remarks

In a discrete time version of Back and Baruch (2004) model with deterministic terminal period, we proved the unique existence of equilibrium. Then, we defined information entropy and showed how manipulation affects the amount of information conveyed to the market. Market price manipulation has been a challenging issue in a market microstructure literature, and one of the difficulties was the dynamic behavior of informed traders. Especially, in the Back and Baruch (2004) version of the Glosten and Milgrom (1985) model, the equilibrium is not tractable. By proving the unique existence of equilibrium in the discrete time with deterministic terminal date, this paper opens up a way to interesting questions in the area.

Obvious extensions of the paper are to consider the infinite-period of the current model with time discount factor and extend it to the continuous time model in order to see if the results still stay. Then, we will be able to see if in the Back and Baruch (2004) model, there exists a unique equilibrium. It is still an open question to see if there exists a unique equilibrium in the Kyle (1985) model. So, in order to answer to the question of "which equilibrium of the Glosten-Milgrom model converges to which equilibrium of the Kyle model?," the issue studied in this paper is important.

As for information entropy, Grossner and Tomala (2008) presented applications to merging theory and to the cost of learning in repeated decision problems. We can apply their method to our model and consider bound on the cost of learning or speed of learning for the market maker. This will give us the implications about how costly market price manipulation is for market makers or liquidity traders. These problems will be interesting directions for future research.

Appendix: Proof of Theorem 1

In order to prove the existence of equilibrium, we consider the equilibrium strategies (σ_L^*, σ_H^*) to be a fixed point of the collection of their best response correspondences $BR = \{BR^t\}_{t=1\cdots,T}$ with $BR^t : [\Delta(A)]^2 \Rightarrow [\Delta(A)]^2$ such that for each t, $(\sigma_L^*, \sigma_H^*) = BR^t(\sigma_L^*, \sigma_H^*)$. Let $U_n^t : \Delta(A) \times [0,1]^2 \to I\!R$ denote the payoff function for the type $n \in N$ trader in period t. More formally, for $n \in \{H, L\}$,

$$U_n^t(\sigma_n, p_t) = \sum_{t'=t}^{T} \left[\sigma_{nB}(\theta - \alpha_{t'}) - \sigma_{nS}(\theta - \beta_{t'})\right]. \tag{67}$$

Then, we define the informed trader's best response correspondence: for every $t \in \{1, \cdots, T\}$ and given p_t,

$$BR^t(\sigma_L, \sigma_H) = \left\{(\sigma_L, \sigma_H) \in [\Delta(A)]^2 | \sigma_n \in \arg\max_{\sigma \in \Delta(A)} U_n^t(\sigma, p_t) \ \ \forall n \in N\right\}. \tag{68}$$

Therefore, when $b(h_t) = b_t$, $\alpha_t^*(b(h_t)) = \alpha_t$ and $\beta_t^*(b(h_t)) = \beta_t$, continuation value of the game for the high-type in period t is:

$$V_H^t(b_t) = \max_{\sigma_H \in \Delta(A)} [\sigma_{HB}(1 - \alpha_t + V_H^{t+1}(b(h_t, B))) + \sigma_{HS}(\beta_t - 1 + V_H^{t+1}(b(h_t, S)))], \tag{69}$$

and one for the low type is:

$$V_L^t(b_t) = \max_{\sigma_L \in \Delta(A)} [-\sigma_{LB}\alpha_t + V_L^{t+1}(b(h_t, B)) + \sigma_{LS}(\beta_t + V_L^{t+1}(b(h_t, S)))]. \tag{70}$$

Thus, an equilibrium defined in Definition 3 is a fixed point of the best response correspondence BR, and α_t and β_t are respectively updated by Bayes rule (1). More formally, we will prove that there exists an fixed point (σ_L^*, σ_H^*) such that: for each $t \in \{1, \cdots, T\}$,

$$BR^t(\sigma_L^*, \sigma_H^*) = (\sigma_L^*, \sigma_H^*). \tag{71}$$

Lemma 12 *The payoff function U_n^t is continuous. In addition, for every t, BR^t is a upper semi-continuous correspondence.*

Proof: Since the argument is symmetric, we only consider the high-type's payoff function and the value function. Note that U_H^t is continuous in his strategy and also the market maker's quotes (β_t, α_t). Then, U_H^t is a continuous numerical function.

We respectively denote the sequences of prices associated with σ^k and $\hat{\sigma}^k$ by p^k and $\hat{p}^k$ and also σ and $\hat{\sigma}$ by p and $\hat{p}$. Then, since the prices are continuous in strategies, we have $p^k \to p$ and $\hat{p}^k \to \hat{p}$.

Now on the contrary, suppose that there exists a sequence as above but $\hat{\sigma} \notin BR(\sigma_H, \sigma_L)$. Without loss of generality, we suppose that there exists a $\epsilon > 0$ and $\bar{\sigma}_H \in \Delta(E)$ such that:

$$U_H^t(\bar{\sigma}_H, p) > U_H^t(\hat{\sigma}_H, p) + 3\epsilon. \tag{72}$$

For k large enough, by continuity of the payoff function and prices, we have:

$$\begin{aligned} U_H^t(\bar{\sigma}_H, p^k) &> U_H^t(\bar{\sigma}_H, p) - \epsilon > U_H^t(\hat{\sigma}_H, p) + 2\epsilon & (73) \\ &> U_H^t(\hat{\sigma}_H^k, p) + \epsilon > U_H^t(\hat{\sigma}_H^k, p^k). & (74) \end{aligned}$$

This contradicts with the fact that $(\hat{\sigma}_H^k, \hat{\sigma}_L^k) \in BR^t(\sigma_H^k, \sigma_L^k)$ for all k. ■

Lemma 13 *The set $[\Delta(A)]^2$ is non-empty, compact and convex.*

Proof: The set of strategies $\Delta(A)$ is non-empty, compact and convex. The set $[\Delta(A)]^2$ is a Cartesian product of those sets and thus the result follows. ■

Lemma 14 *The informed trader's best response correspondence BR^t is non-empty and convex-valued for every $t \in \{1, \cdots, T\}$.*

Proof: We will prove this by mathematical induction. Since the argument is symmetric, we only consider the high type. Consider the last period $t = T$. Then, the high type and low type trade on their information. In this sense, BR^T is non-empty and convex-valued. Next we suppose that in period $t+1$, BR^{t+1} is non-empty and convex-valued. Then, we will prove that in period t, BR^t is also non-empty and convex-valued.

By the assumption for the inductive hypothesis, we know that V_H^{t+1} is well-defined. Now, fix a history h^{t-1} arbitrarily. Then, given V_H^{t+1}, the right hand side of the expression in (69) is linear in the strategies σ_H. Therefore the expression in (69) has a maximum so that the set BR^t is non-empty.

Second, we will prove that it is also convex-valued. Take two different strategies $(\bar{\sigma}_H, \bar{\sigma}_L) \in BR^t(\bar{\sigma}_H, \bar{\sigma}_L)$ and $(\bar{\bar{\sigma}}_H, \bar{\sigma}_L) \in BR^t(\bar{\sigma}_H, \bar{\sigma}_L)$. We denote the prices associated with the strategies $(\bar{\sigma}_H, \bar{\sigma}_L)$ by $\bar{p}_t$. Then, the following must hold:

$$U_H^t(\bar{\sigma}_H, \bar{p}_t) = U_H^t(\bar{\bar{\sigma}}_H, \bar{p}_t). \tag{75}$$

Let $\hat{\sigma}_H^t = \gamma\bar{\sigma}_H + (1-\gamma)\bar{\bar{\sigma}}_H$ for some $\gamma \in (0,1)$. By using linearity of the payoff function, we have:

$$U_H^t(\hat{\sigma}_1, \bar{p}_t) \quad = \quad \gamma U_H^t(\bar{\sigma}_H, \bar{p}_t) + (1-\gamma)U_H^t(\bar{\bar{\sigma}}_H, \bar{p}_t) = U_H^t(\bar{\sigma}_H, \bar{p}_t) = U_H^t(\bar{\bar{\sigma}}_H, \bar{p}_t). \tag{76}$$

and therefore we have: $(\hat{\sigma}_H, \bar{\sigma}_L) \in BR^t(\bar{\sigma}_H, \bar{\sigma}_L)$. Therefore, BR^t is convex-valued. ■

Proof of Theorem 1: By Lemma 12 to Lemma 14, we can apply the Kakutani's fixed point theorem to the best response correspondence BR^t on $[\Delta(A)]^2$ for all $t \in \{1, \cdots, T\}$. ■

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