IN SEARCH OF AN OPTIMISING AGENT WITH CYCLICAL BEHAVIOUR

by

Peter J. Stemp*

Economics Program
Research School of Social Sciences
Australian National University
Canberra, ACT, 0200
Australia
email: peter.stemp@anu.edu.au

Revised Version
December, 2007

* The first draft of this paper was completed while the author was a visitor in the Department of Economics at the University of Washington in Seattle. Subsequent versions were presented at the Macroeconomics Workshop at the University of Melbourne and at the Operations Research (ORSUM) Seminar at the University of Melbourne as well as to the Society of Computational Economics, 11th International Conference on Computing in Economics and Finance, Washington, D.C., June 23-25, 2005 and to the 11th Australasian Macroeconomics Workshop, H. C. Coombs Centre, Kirribili, Sydney, April 20-21, 2006. The author is especially grateful to Suren Basov, Bruce Craven, Ian Enting, Nisvan Erkal, Ian McDonald, Sir James Mirrlees, Nilss Olekalns, Jeffrey Sheen, John Stachurski, Stephen Turnovsky and to other participants at these presentations for helpful comments. The author is also grateful to the University of Washington for hospitality and support and to the Faculty of Economics and Commerce at the University of Melbourne for financial assistance through a Faculty Research Grant.

ABSTRACT

Most recent studies of dynamic macroeconomic relationships focus on models derived from optimising behaviour by economic agents. In most of these models, the eigenvalues of the associated dynamical system are real-valued and so the time-path of the system exhibits monotonic or near-monotonic behaviour. While, it is well-known that linear dynamic models with complex-valued eigenvalues exhibit the property of oscillatory dynamic behaviour, limited research has been undertaken to investigate the properties of optimising models with oscillatory behaviour. In this study, we produce an example of an optimising consumer with habit persistence whose consumption dynamics are characterised by complex-valued eigenvalues and whose consumption thus exhibits cyclical behaviour. In practice, even in this case, oscillatory behaviour will not be observable if the periodicity of the cycle is too long or if the cycle becomes dampened at too rapid a rate. For example, there will be little evidence of cycles for a consumer who lives for eighty years if they have a consumption cycle with a period-length of two hundred years. In order to investigate whether cycles are likely to be empirically observable, we also investigate the dynamic properties of a calibrated version of the model.

JEL classification: E21; E27.

Keywords: Macroeconomics; Dynamical Systems; Complex-valued Eigenvalues; Consumption; Habit Persistence.

1. INTRODUCTION

Recent studies of dynamic macroeconomic relationships focus on models derived from optimising behaviour by economic agents. In most of these models, the eigenvalues of the associated dynamical system are real-valued and so the time-path of the system exhibits monotonic or near-monotonic behaviour (see, for example, Blanchard and Fischer, 1989, and Turnovsky, 2000). While it is well-known that linear dynamic models with complex-valued eigenvalues exhibit the property of oscillatory dynamic behaviour (see, for example, Simon and Blume, 1994), limited research has been undertaken to investigate the properties of optimising models with oscillatory behaviour.

In this study, we take the most basic model of the consumer, the model of optimal saving attributed to Ramsey (1928) and modify the model to allow for habit persistence in consumption. The consumption behaviour of the optimising consumer is then derived using the standard techniques of optimal control theory (Leonard and Long, 1992). Applying these techniques, we are able to show that it is possible for the linearised version of this model to have complex-valued eigenvalues. The model can thus exhibit cyclical behaviour for an appropriate choice of model parameters.

Of course, in practice, such cyclical behaviour is unimportant if the period length of the cycles is too long (for example, if the consumer lives for 80 years but the length of the cycles is 100 years) or if the model has substantially converged to its steady-state well within the length of a full period. To investigate whether it is possible to generate meaningful cycles we then calibrate the model, using plausible parameter values.

The rest of the paper proceeds as follows. Section 2 introduces the basic model. The model's dynamic properties are investigated in Section 3. After specific functional forms for the utility and production functions have been introduced in Section 4, the model is calibrated and results reported in Section 5. Section 6 summarises our conclusions.

2. THE MODEL

Consider the following problem for a representative consumer with habit persistence in consumption:

$$\underset{c}{\text{Max}}\ V = \int_0^\infty \exp(-\delta t)[u(c) - \tfrac{1}{2}\alpha(\dot{c})^2]dt \tag{1}$$

subject to:

$$\dot{k} = f(k) - (\rho + n)k - c \tag{2}$$

where

k = capital/labour ratio;

c = consumption/labour ratio;

δ = discount rate;

ρ = rate of capital depreciation; and

n = rate of population growth.

It is also assumed that $u' > 0, u'' < 0$, $f' > 0$ and $f'' < 0$.

The chosen model is the Ramsey (1928) model of optimal saving with an additional term in the criterion function used to model habit persistence. This additional term is given by: $"-\tfrac{1}{2}\alpha\left(\dot{c}\right)^2"$.

We can rewrite equation (2) as

$$\dot{k} = z \tag{3a}$$

$$z = f(k) - (\rho + n)k - c \tag{3b}$$

and derive the following equation for $\dot{c}$:

$$\dot{c} = f'(k)z - (\rho + n)z - \dot{z} \tag{3c}$$

Then, to solve the consumer's problem, we first write down the Euler-Lagrange Hamiltonian as follows:

$$H = \exp(-\delta t)\{u(c) - \tfrac{1}{2}\alpha[f'(k)z - (\rho+n)z - \dot{z}]^2\} + \exp(-\delta t)\psi[z - \dot{k}] +$$

$$\exp(-\delta t)\eta[z - f(k) + (\rho+n)k + c] \tag{4}$$

where ψ and λ are discounted co-state variables.

The Euler-Lagrange conditions for this Hamiltonian then satisfy:

$$H_c = 0 \tag{5a}$$

$$H_z = \frac{d}{dt}H_{\dot{z}} \tag{5b}$$

$$H_k = \frac{d}{dt}H_{\dot{k}} \tag{5c}$$

These equations then reduce to:

$$u'(c) + \eta = 0 \tag{6a}$$

$$\alpha\ddot{c} + \alpha\dot{c}[f'(k) - \rho - n - \delta] - \psi - \eta = 0 \tag{6b}$$

$$\dot{\psi} - \delta\psi - \alpha\dot{c}[f''(k)z] - \eta f'(k) + \eta(\rho+n) = 0 \tag{6c}$$

In turn, these equations can be written as the following four-dimensional equation system, with endogenous variables given by: *c*, *k*, ψ and *x*.

$$\dot{c} = x \tag{7a}$$

$$\dot{k} = f(k) - (\rho+n)k - c \tag{7b}$$

$$\dot{\psi} = \delta\psi + \alpha x\{f''(k)[f(k) - (\rho+n)k - c]\} - u'(c)[f'(k) - \rho - n] \tag{7c}$$

$$\dot{x} = x[\rho + n + \delta - f'(k)] + \frac{1}{\alpha}[\psi - u'(c)] \tag{7d}$$

The steady state of this system satisfies the following equations:

$$c^* = f(k^*) - (\rho + n)k^* \tag{8a}$$

$$f'(k^*) = \rho + n + \delta \tag{8b}$$

$$\psi^* = u'(c^*) \tag{8c}$$

$$x^* = 0 \tag{8d}$$

Then the system can be linearized about its steady-state yielding:

$$\begin{pmatrix} \dot{c} \\ \dot{k} \\ \dot{\psi} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & \delta & 0 & 0 \\ -u''(c^*)\delta & -u'(c^*)f''(k^*) & \delta & 0 \\ -\frac{u''(c^*)}{\alpha} & 0 & \frac{1}{\alpha} & 0 \end{pmatrix} \begin{pmatrix} c - c^* \\ k - k^* \\ \psi - \psi^* \\ x - x^* \end{pmatrix} \tag{9}$$

3. ESTABLISHING DYNAMIC PROPERTIES OF MODEL

Calculating eigenvalues

In order to solve the model, it is first necessary to calculate the eigenvalues.

The characteristic equation satisfies:

$$0 = c(\lambda) = \begin{vmatrix} -\lambda & 0 & 0 & 1 \\ -1 & \delta - \lambda & 0 & 0 \\ -u''(c^*)\delta & -u'(c^*)f''(k^*) & \delta - \lambda & 0 \\ -\frac{u''(c^*)}{\alpha} & 0 & \frac{1}{\alpha} & -\lambda \end{vmatrix} \tag{10a}$$

$$= \lambda^2(\delta - \lambda)^2 + (\delta - \lambda)^2 \frac{u''(c^*)}{\alpha} + (\delta - \lambda)\frac{u''(c^*)\delta}{\alpha} - \frac{u'(c^*)f''(k^*)}{\alpha} \tag{10b}$$

$$= \alpha\lambda^2(\delta - \lambda)^2 + (\delta - \lambda)^2 u''(c^*) + (\delta - \lambda)u''(c^*)\delta - u'(c^*)f''(k^*) \tag{10c}$$

Letting

$$\kappa = -\frac{\alpha}{u''(c^*)} \tag{11a}$$

and $$\theta = \frac{u'(c^*) f''(k^*)}{u''(c^*)} \tag{11b}$$

Equation (10c) can be rewritten as:

$$c(\lambda) = \kappa\lambda^2(\delta-\lambda)^2 - (\delta-\lambda)^2 - (\delta-\lambda)\delta + \theta = 0 \tag{12}$$

Proposition 1: *When $\delta \to 0$ from above, we can always choose a value of κ (and hence a value of α) so that all eigenvalues of the model given by equation (9) are complex-valued.*

Proof:

Let $\delta \to 0$ from above in equation (12). Then the characteristic equation reduces to:

$$c(\lambda) = \kappa\lambda^4 - \lambda^2 + \theta = 0 \tag{13}$$

The eigenvalues, as $\delta \to 0$, are then given by:

$$\lambda^2 = \frac{1 \pm \sqrt{1-4\kappa\theta}}{2\kappa} \tag{14}$$

Let $\Delta = \dfrac{1-4\kappa\theta}{4\kappa^2}$, then $\Delta = 0$ when $\kappa = \overline{\kappa} = \dfrac{1}{4\theta}$.

Then,

$$\lambda^2 \text{ is real-valued} \Leftrightarrow \Delta > 0 \Leftrightarrow \kappa < \overline{\kappa} \tag{15a}$$

$$\lambda^2 \text{ is complex-valued} \Leftrightarrow \Delta < 0 \Leftrightarrow \kappa > \overline{\kappa} \tag{15b}$$

At $\kappa = \overline{\kappa}$,

$$\lambda^2 = \pm 2\theta = \pm \frac{2u'(c^*) f''(k^*)}{u''(c^*)} \tag{16}$$

So that

$$\lambda_1 = \lambda_3 = +\sqrt{2\theta} = +\sqrt{\frac{2u'(c^*)f''(k^*)}{u''(c^*)}} \tag{17a}$$

$$\lambda_2 = \lambda_4 = -\sqrt{2\theta} = -\sqrt{\frac{2u'(c^*)f''(k^*)}{u''(c^*)}} \tag{17b}$$

Furthermore, Δ is minimized when $\kappa = \hat{\kappa}$ and $\Delta = \hat{\Delta}$, with $\hat{\kappa}$ and $\hat{\Delta}$ given by:

$$\hat{\kappa} = \frac{1}{2\theta} \text{ and } \hat{\Delta} = -\theta^2 \tag{17}$$

Figure 1 plots Δ against κ.

(Figure 1 about here)

Thus, as $\delta \to 0$, λ^2 has its largest imaginary part when $\alpha = \hat{\alpha}$ and $\Delta = \hat{\Delta}$. At that point,

$$\lambda^2 = \theta(1 \pm i) = \left(\frac{u'(c^*)f''(k^*)}{u''(c^*)}\right)(1 \pm i) \tag{18a}$$

But,

$$1 + i = (\sqrt{2})\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] = (\sqrt{2})\left[\exp\left(\frac{i\pi}{4}\right)\right] \tag{19a}$$

$$1 - i = (\sqrt{2})\left[\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right] = (\sqrt{2})\left[\exp\left(\frac{i7\pi}{4}\right)\right] \tag{19b}$$

Then, the four eigenvalues of the system are given by:

$$\lambda_1, \lambda_2 = \pm\sqrt{\left(\frac{u'(c^*)f''(k^*)}{u''(c^*)}\right)(1+i)}$$

$$= \pm\left(\sqrt{\frac{u'(c^*)f''(k^*)}{u''(c^*)}}\right)(\sqrt[4]{2})\exp\left(\frac{i\pi}{8}\right) \tag{20a}$$

$$\lambda_3, \lambda_4 = \pm\sqrt{\left(\frac{u'(c^*)f''(k^*)}{u''(c^*)}\right)(1-i)}$$

$$= \pm \left(\sqrt{\frac{u'(c^*) f''(k^*)}{u''(c^*)}} \right) (\sqrt[4]{2}) \exp\left(\frac{i7\pi}{8} \right) \qquad (20b)$$

For each pair of equations (20a-20b), one complex-valued eigenvalue has positive real part and one has negative real part. Thus there is total of two stable eigenvalues and two unstable eigenvalues. ***End of Proof.***

Closed-form solution of model

Restricting our analysis to the case when all eigenvalues are complex-valued, we will write the stable eigenvalues as $\lambda_1 (= -\alpha + i\beta)$ and $\lambda_2 (= -\alpha - i\beta)$ and the unstable eigenvalues as $\lambda_3 (= \gamma + i\varepsilon)$ and $\lambda_4 (= \gamma - i\varepsilon)$, where α, β, γ and ε are positive real-valued constants. For each eigenvalue we can then use equation (9) to calculate the associated eigenvectors which, for each eigenvalue, λ, are given by:

$$\mathbf{v}(\lambda) = \begin{pmatrix} \delta - \lambda \\ 1 \\ (\delta - \lambda)[\alpha\lambda^2 + u''(c^*)] \\ \lambda(\delta - \lambda) \end{pmatrix} \qquad (21)$$

Then, the general closed-form solution of the model is given by:

$$\begin{pmatrix} c - c^* \\ k - k^* \\ \psi - \psi^* \\ x - x^* \end{pmatrix} = \begin{bmatrix} \mathbf{v}(\lambda_1) & \mathbf{v}(\lambda_2) & \mathbf{v}(\lambda_3) & \mathbf{v}(\lambda_4) \end{bmatrix} \begin{bmatrix} (A_1 + iA_2)\exp(\lambda_1 t) \\ (A_1 - iA_2)\exp(\lambda_2 t) \\ (B_1 + iB_2)\exp(\lambda_3 t) \\ (B_1 - iB_2)\exp(\lambda_4 t) \end{bmatrix} \qquad (22)$$

where there are two "jump" variables, ψ and $x(=\dot{c})$, so that the constants: A_1, A_2, B_1 and B_2 are determined by initial values for *c* and *k*, and by the transversality conditions.

Since we are primarily interested in the dynamic properties of *c* and *k*, we can use equation (22) to yield the following solutions for *c* and *k*:

$$c - c^* = \exp(-\alpha t)[2\{A_1(\delta+\alpha) + A_2\beta\}\cos(\beta t) + 2\{A_1\beta - A_2(\delta+\alpha)\}\sin(\beta t)]$$
$$+\exp(\gamma t)[2\{B_1(\delta-\gamma) + B_2\varepsilon\}\cos(\varepsilon t) + 2\{B_1\varepsilon - B_2(\delta-\gamma)\}\sin(\varepsilon t)] \quad (23a)$$

$$k - k^* = \exp(-\alpha t)[2A_1\cos(\beta t) - 2A_2\sin(\beta t)]$$
$$+\exp(\gamma t)[2B_1\cos(\varepsilon t) - 2B_2\sin(\varepsilon t)] \quad (23b)$$

4. CHOOSING SPECIFIC FUNCTIONAL FORMS

We now investigate the values of the eigenvalues generated by equation (12) using specific functional forms. In particular we need to establish a formula to determine the magnitude of: $\theta\left(=\dfrac{u'(c^*)f''(k^*)}{u''(c^*)}\right)$. To do this, we assume that:

$$f(k) = ak^{\beta} \quad (24a)$$

and $$u(c) = \frac{c^{1-\eta}}{1-\eta} \quad (24b)$$

Hence,

$$f''(k) = \beta(\beta-1)ak^{\beta-2} \quad (25a)$$

Also,

$$\frac{u'(c)}{u''(c)} = -\frac{c}{\eta} \quad (25b)$$

So that

$$\theta = \frac{u'(c^*)f''(k^*)}{u''(c^*)} = \frac{\beta(1-\beta)a(k^*)^{\beta-2}c^*}{\eta} \tag{26}$$

Also, substituting equations (24a-24b) into equations (8a-8b) yields

$$c^* = a(k^*)^{\beta} - (\rho+n)k^* \tag{27a}$$

$$\beta a(k^*)^{\beta-1} = \rho + n + \delta \tag{27b}$$

From equation (27b),

$$k^* = \left(\frac{\rho+n+\delta}{\beta a}\right)^{\frac{1}{\beta-1}} \tag{27c}$$

Substituting equation (27a) into equation (26) yields

$$\theta = \frac{\beta(1-\beta)[a^2(k^*)^{2\beta-2} - (\rho+n)a(k^*)^{\beta-1}]}{\eta} \tag{28a}$$

Then substituting equation (27c) into equation (27a) yields

$$\theta = \frac{\beta(1-\beta)}{\eta}\left[a^2\left(\frac{\rho+n+\delta}{\beta a}\right)^2 - (\rho+n)a\left(\frac{\rho+n+\delta}{\beta a}\right)\right] \tag{28b}$$

$$= \frac{\beta(1-\beta)}{\eta}\left[\frac{(\rho+n+\delta)^2}{\beta^2} - \frac{(\rho+n)(\rho+n+\delta)}{\beta}\right] \tag{28c}$$

$$= \frac{(1-\beta)(\rho+n+\delta)}{\beta\eta}\left[(\rho+n+\delta) - \beta(\rho+n)\right] \tag{28d}$$

In the special case, when $\delta \to 0$, considered above, this reduces further to:

$$\theta = \frac{u'(c^*)f''(k^*)}{u''(c^*)} = \frac{(1-\beta)^2(\rho+n)^2}{\beta\eta} \tag{29}$$

Note that, from equations (11a), (24b) and (25b),

$$\kappa = -\frac{\alpha}{u''(c^*)} = \frac{\alpha}{\eta}c^{\eta+1} \tag{30}$$

But, from equations (27a-27c), the magnitude of c^* is determined by the magnitude of a. Hence, for a given κ, the precise magnitude of α is determined by the magnitude of a.

5. CALIBRATING THE MODEL

We shall now calibrate the model by assuming $\beta = 0.25$, implying a capital share of income of 25%, instantaneous intertemporal elasticity of substitution, $1/\eta = 1.0$ and population growth rate, $n = 0.02$. As demonstrated in equations (28d, 29), the value taken by a is irrelevant for the determination of θ. We also consider three rates of capital depreciation, given by $\rho = 0.10$, $\rho = 0.20$ and $\rho = 0.30$ as well as three discount rates given by $\delta = 0.00$, $\delta = 0.10$ and $\delta = 0.20$.

Our process for deriving eigenvalues from equation (12) is to determine θ and δ using the calibrated parameter values and then to investigate the properties of the eigenvalue configurations as κ is allowed to vary through arrange of values. All reported results have been derived using Mathematica version 5.1.1 (Wolfram, 2003).

(Table 1 about here)

Table 1 gives the values of θ and $\hat{\kappa}$ for $\rho = 0.10$, $\rho = 0.20$ and $\rho = 0.30$ under the assumption that $\delta = 0$. We have shown that, when $\delta = 0$, the absolute imaginary parts of λ^2 are maximized when $\kappa = \hat{\kappa}$. Usually, this does not mean that the absolute imaginary parts of each eigenvalue are maximized for $\hat{\kappa}$ but it does give an indication of where to locate our grid-search for finding optimal values of κ.

In order to investigate the oscillatory properties of solutions with complex-valued eigenvalues, it is appropriate to rewrite equations (23a-23b):

$$c - c^* = \exp(-\alpha t)[2\{A_1(\delta+\alpha) + A_2\beta\}\cos(\beta t) + 2\{A_1\beta - A_2(\delta+\alpha)\}\sin(\beta t)]$$

$$+\exp(\gamma t)[2\{B_1(\delta-\gamma)+B_2\varepsilon\}\cos(\varepsilon t)+2\{B_1\varepsilon-B_2(\delta-\gamma)\}\sin(\varepsilon t)]$$

(23a)

$$k-k^*=\exp(-\alpha t)[2A_1\cos(\beta t)-2A_2\sin(\beta t)]$$

$$+\exp(\gamma t)[2B_1\cos(\varepsilon t)-2B_2\sin(\varepsilon t)] \qquad (23b)$$

Following an unanticipated shock, all variables will jump to the stable manifold where $B_1 = B_2 = 0$, so it will be appropriate to examine the oscillatory properties for c and k when solutions are restricted solely along the stable manifold. Following an anticipated shock, that is a shock that is announced at time t_0 but not implemented until some later time $t_1(> t_0)$, then the economic variables will spend some time following an unstable path, before reaching the stable manifold at time t_1. To examine this case it is appropriate to consider solutions along the unstable manifold where $A_1 = A_2 = 0$.

Solutions along the stable manifold

(Tables 2A and 2B about here)

We first consider solutions along the stable manifold. Table 2A uses a grid-search over a range of κ values to find the approximate eigenvalue configurations that have maximum absolute imaginary part for the stable eigenvalues. These will also be the configurations that yield the smallest period length if the solution is restricted to the stable manifold. Column 3 of Table 2B shows the period length in years for the stable eigenvalues, where the stable eigenvalues are given by $-\alpha \pm i\beta$ and the period length is given by the formula:

$$\text{Period length } = ps = \frac{2\pi}{\beta} \qquad (31a)$$

Of course, we will not have observable behaviour if the amplitude of the cycle has diminished to almost zero well within the length of a full period. To investigate this outcome we calculate the amplitude after one period along the stable manifold assuming that the amplitude is equal to one at time zero. Amplitude along the stable manifold is given by the formula:

$$\text{Amplitude } = as = \exp\left(\frac{-\alpha.2\pi}{\beta}\right) \tag{31b}$$

The amplitude associated with stable eigenvalues is reported in Column 4 of Table 2B.

In the case of unstable eigenvalues, we need to calculate the amplitude along the unstable manifold. Since movement along the unstable eigenvalue is associated with an anticipated shock, it is appropriate to consider the amplitude after a period of five years, since shocks are unlikely to be anticipated more than five years ahead of time. Column 5 of Table 2B shows the period length in years for the unstable eigenvalues, where the unstable eigenvalues are given by $\gamma \pm i\varepsilon$ and the period length is given by the formula:

$$\text{Period length } = pu = \frac{2\pi}{\varepsilon} \tag{32a}$$

The amplitude associated with unstable eigenvalues is reported in Column 6 of Table 2B and satisfies the formula:

$$\text{Amplitude } = au = \exp\left(\frac{10\pi}{\varepsilon}\right) \tag{32b}$$

Examination of Tables 2A and 2B indicates that the largest period lengths in the case examined arise for the largest values of $\delta(=0.20)$ and $\rho(=0.30)$. Yet even in this case the period lengths are substantial (21.1 years for stable eigenvalues and 13.8 years for unstable eigenvalues). Also along the stable manifold the solutions will have converged to close to their steady-state well before the end of the period. These results indicate that observable cyclical behaviour is unlikely to be achieved for any of the chosen parameter values.

(Figures 2A and 2B about here)

These results are confirmed by Figures 2A and 2B which examine time-paths of c and k along the stable manifold (in Figure 2A) and the unstable manifold (in Figure 2B) for a range of initial conditions when $\delta = 0.20$ and $\rho = 0.30$. For all initial conditions, there is no observable oscillatory behaviour, although the occasional hump does arise.

Solutions along the unstable manifold

It is still possible that oscillatory behaviour might arise along the unstable manifold if the imaginary part of unstable eigenvalues were maximised. This case, which is associated with minimum period lengths along the unstable manifold, is considered in Tables 3A and 3B. Once again a reduced minimum period length along the unstable manifold arises when $\delta = 0.20$ and $\rho = 0.30$ but the reduction is not significant. These results are confirmed by Figure 3A which examines time-paths of c and k along the unstable manifold for a range of initial conditions when $\delta = 0.20$ and $\rho = 0.30$. Once again, for all initial conditions, there is no observable oscillatory behaviour, although the occasional hump does arise. Again we conclude that cyclical behaviour is unlikely to be observed along the unstable manifold for any of the chosen parameter values.

(Tables 3A and 3B about here)

(Figure 3A about here)

6. CONCLUSION

This paper has investigated whether or not it is possible to generate oscillatory behaviour in a standard model of consumer behaviour. The study has been undertaken by extending the Ramsey (1928) model to allow for habit persistence in consumption. Our results show that, using this approach, it is possible to generate dynamic behaviour which is characterised by complex-valued eigenvalues. Hence oscillatory behaviour is theoretically possible. However, when the model was calibrated with plausible parameter values, the results provide overwhelming evidence that observable oscillations are unlikely to occur in practice.

The results indicate that it is likely that observable cyclical behaviour will only be able to be generated using considerably more complicated models of consumer behaviour than were considered here.

REFERENCES

Blanchard, Olivier J., and Stanley Fischer (1989), *Lectures on Macroeconomics*, MIT Press, Cambridge Massachusetts and London England.

Leonard, Daniel, and Ngo Van Long (1992), *Optimal Control Theory and Static Optimization in Economics*, Cambridge University Press, Cambridge England.

Ramsey, Frank, P. (1928), "A Mathematical Theory of Saving," *Economic Journal*, 38, No. 152 (December), pp. 543-559. Reprinted in Joseph E. Stiglitz and Hirofumi Uzawa (editors), *Readings in the Modern Theory of Economic Growth*, MIT Press, 1969.

Simon, Carl P., and Lawrence Blume (1994), *Mathematics for Economists*, W. W. Norton and Company, New York and London.

Turnovsky, Stephen J. (2000), *Methods of Macroeconomic Dynamics*, Second Edition, MIT Press, Cambridge Massachusetts and London England.

Wolfram, Stephen (2003), *Mathematica Book*, Fifth Edition, Wolfram Media, Champaign, Illinois.

Figure 1
Plot of Δ against κ when $\delta = 0$

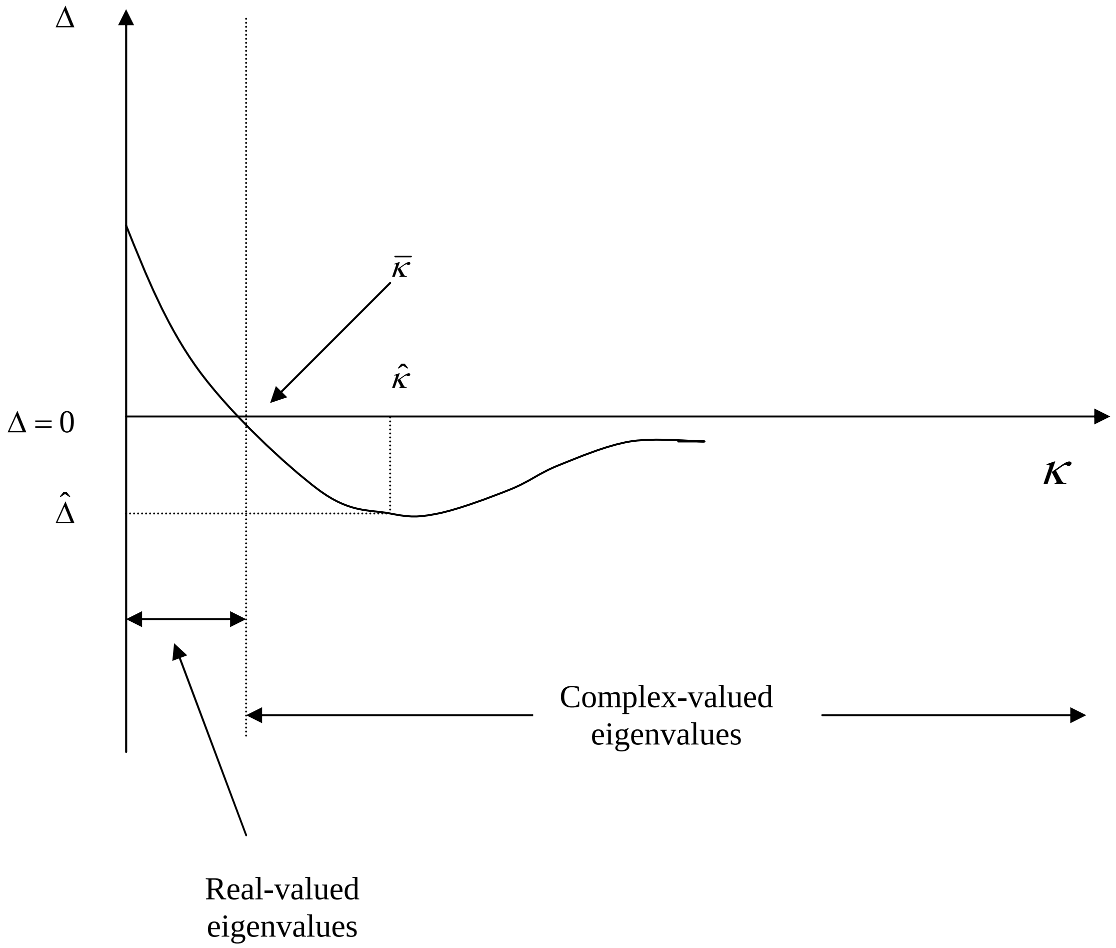


Table 1
Values of θ and $\hat{\kappa}$ for given values of ρ when $\delta = 0.0$

	Values of ρ		
	0.10	0.20	0.30
θ	0.0324	0.1089	0.2304
$\hat{\kappa}$	15.4321	4.5914	2.1701

Table 2A
Eigenvalues with Largest Imaginary Parts for Stable Eigenvalues

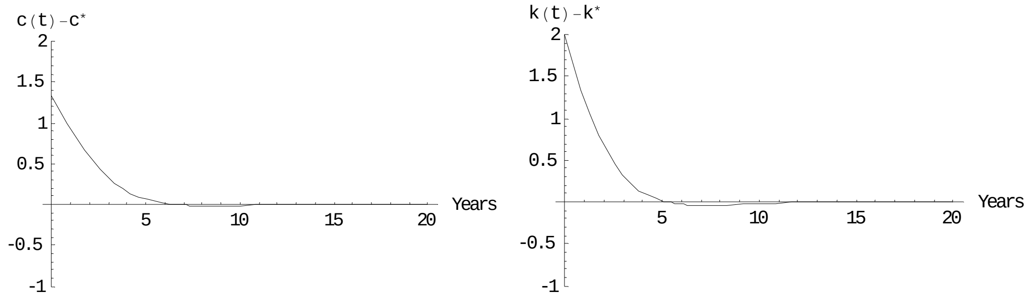
δ	ρ	κ	Stable Eigenvalues	Unstable Eigenvalues
0.0	0.1	30	$-0.1574 \pm 0.0900i$	$+0.1574 \pm 0.0900i$
	0.2	15	$-0.2435 \pm 0.1610i$	$+0.2435 \pm 0.1610i$
	0.3	5	$-0.3176 \pm 0.2256i$	$+0.3176 \pm 0.2256i$
0.1	0.1	20	$-0.1792 \pm 0.1189i$	$+0.2792 \pm 0.1896i$
	0.2	10	$-0.2743 \pm 0.1931i$	$+0.3743 \pm 0.2610i$
	0.3	4	$-0.4259 \pm 0.2697i$	$+0.5259 \pm 0.3539i$
0.2	0.1	15	$-0.1953 \pm 0.1458i$	$+0.3953 \pm 0.2577i$
	0.2	4	$-0.3510 \pm 0.2214i$	$+0.5510 \pm 0.3711i$
	0.3	3	$-0.4693 \pm 0.2981i$	$+0.6693 \pm 0.4539i$

Table 2B
Period Length and Amplitude for Eigenvalues from Table 2A

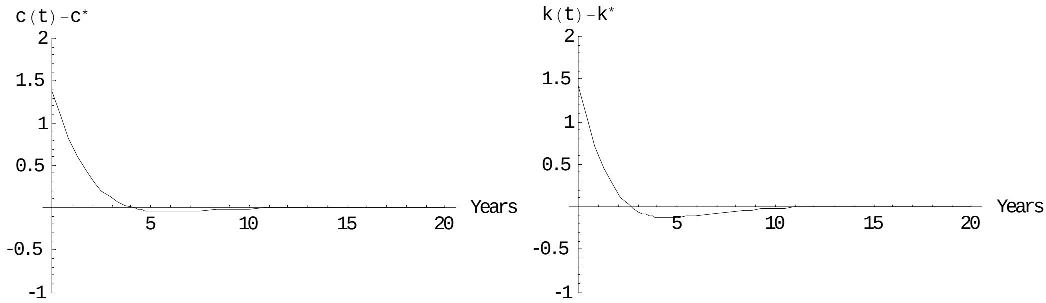
δ	ρ	Stable Eigenvalues		Unstable Eigenvalues	
		Period Length (in years)	Amplitude (after one period)	Period Length (in years)	Amplitude (after five years)
0.0	0.1	69.8	0.000017	69.8	2.197
	0.2	39.0	0.000075	39.0	3.379
	0.3	27.9	0.000144	27.9	4.894
0.1	0.1	52.8	0.000077	33.1	4.039
	0.2	32.5	0.000133	24.1	6.498
	0.3	23.3	0.000049	17.8	13.867
0.2	0.1	43.1	0.000221	24.4	7.217
	0.2	28.4	0.000047	16.9	15.721
	0.3	21.1	0.000051	13.8	28.403

Figure 2A
Plots of *c* and *k* on Stable Manifold for Eigenvalues in Table 2A
When $\delta = 0.2$ and $\rho = 0.3$.

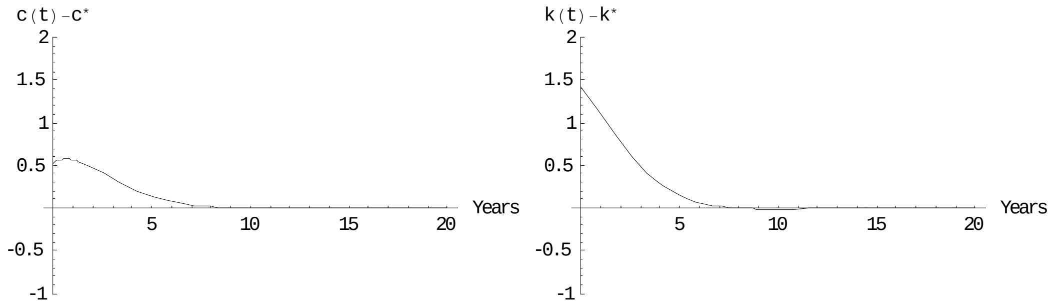
$A_1 = 1, A_2 = 0, B_1 = 0, B_2 = 0$



$A_1 = \frac{1}{\sqrt{2}}, A_2 = \frac{1}{\sqrt{2}}, B_1 = 0, B_2 = 0$



$A_1 = \frac{1}{\sqrt{2}}, A_2 = \frac{-1}{\sqrt{2}}, B_1 = 0, B_2 = 0$



$A_1 = 0, A_2 = 1, B_1 = 0, B_2 = 0$

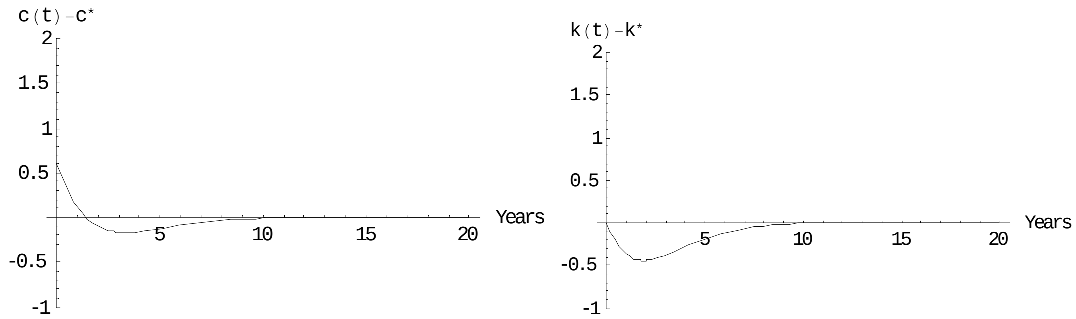
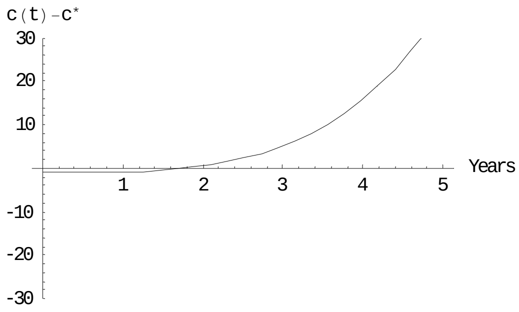
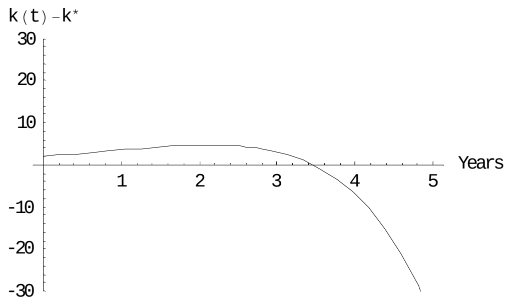


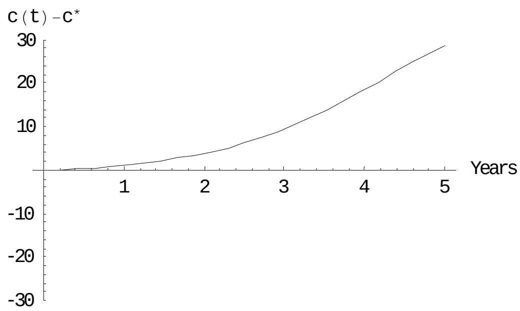
Figure 2B
Plots of *c* and *k* on Unstable Manifold for Eigenvalues in Table 2A
When $\delta = 0.2$ and $\rho = 0.3$.

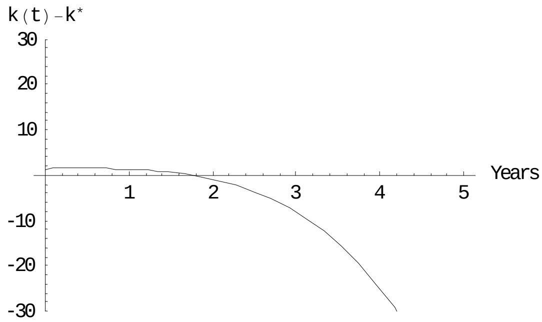
$$A_1 = 0, A_2 = 0, B_1 = 1, B_2 = 0$$



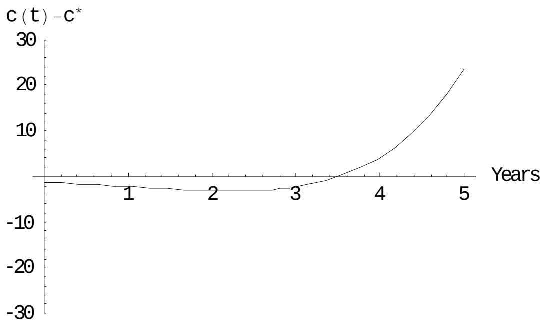


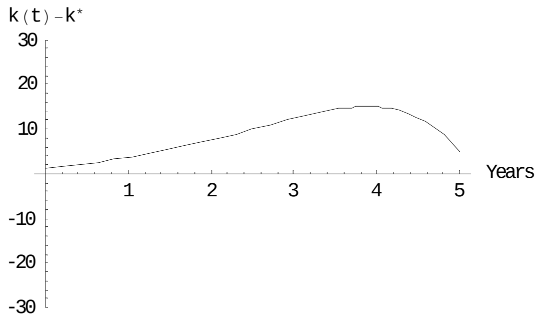
$$A_1 = 0, A_2 = 0, B_1 = \tfrac{1}{\sqrt{2}}, B_2 = \tfrac{1}{\sqrt{2}}$$



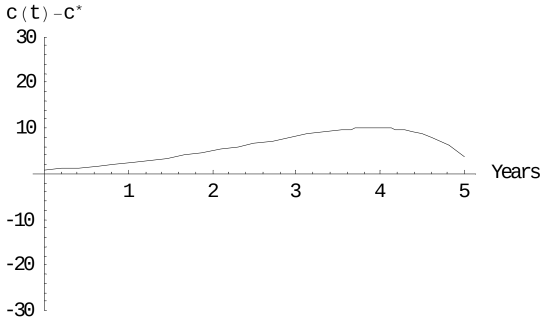


$$A_1 = 0, A_2 = 0, B_1 = \tfrac{1}{\sqrt{2}}, B_2 = \tfrac{-1}{\sqrt{2}}$$





$$A_1 = 0, A_2 = 0, B_1 = 0, B_2 = 1$$



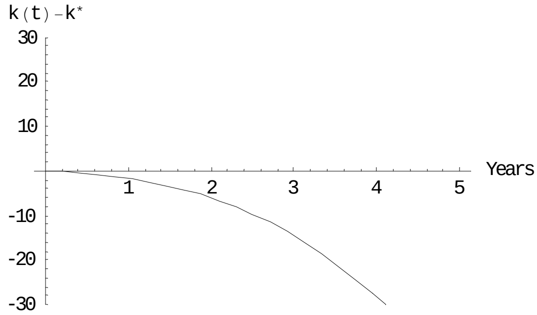


Table 3A
Eigenvalues with Largest Imaginary Parts for Unstable Eigenvalues

δ	ρ	κ	Stable Eigenvalues	Unstable Eigenvalues
0.0	0.1	30	$-0.1574 \pm 0.0900i$	$+0.1574 \pm 0.0900i$
	0.2	15	$-0.2435 \pm 0.1610i$	$+0.2435 \pm 0.1610i$
	0.3	5	$-0.3176 \pm 0.2256i$	$+0.3176 \pm 0.2256i$
0.1	0.1	4	-0.4313 and -0.2462	$0.4387 \pm 0.2361i$
	0.2	2	$-0.5008 \pm 0.7419i$	$0.6008 \pm 0.3103i$
	0.3	2	$-0.5471 \pm 0.2494i$	$0.6471 \pm 0.3820i$
0.2	0.1	1	-0.9852 and -0.2485	$0.8168 \pm 0.3786i$
	0.2	1	-0.8935 and -0.4418	$0.8676 \pm 0.4578i$
	0.3	0.8	-0.8116 and -0.7502	$0.9809 \pm 0.5318i$

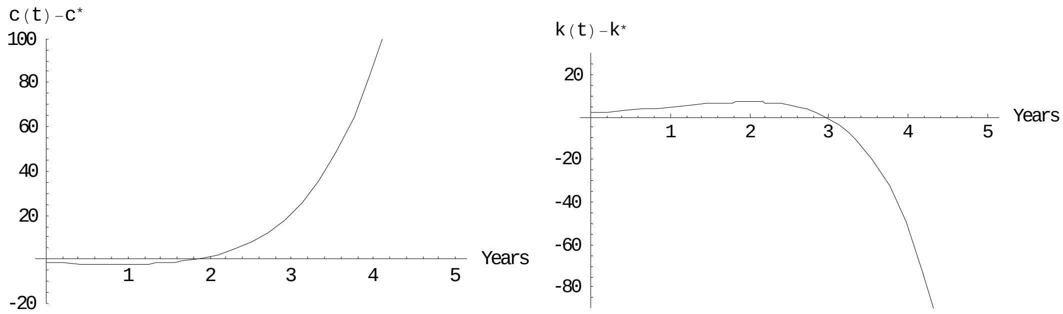
Table 3B
Period Length and Amplitude for Eigenvalues from Table 3A

δ	ρ	Stable Eigenvalues		Unstable Eigenvalues	
		Period Length (in years)	Amplitude (after one period)	Period Length (in years)	Amplitude (after five years)
0.0	0.1	69.8	0.000017	69.8	2.197
	0.2	39.0	0.000075	39.0	3.379
	0.3	27.9	0.000144	27.9	4.894
0.1	0.1	*	*	26.6	8.967
	0.2	8.5	0.014389	20.2	20.166
	0.3	25.2	0.000001	16.5	25.419
0.2	0.1	*	*	16.6	59.383
	0.2	*	*	13.7	76.554
	0.3	*	*	11.8	134.895

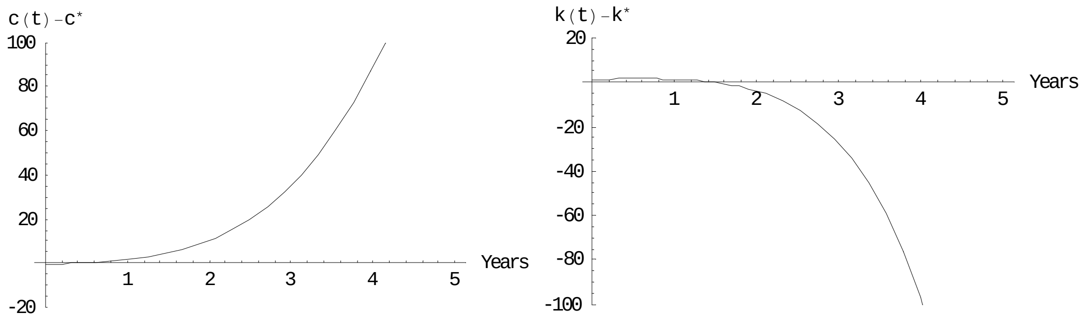
* denotes real-valued eigenvalues so that periodicity is ∞.

Figure 3A
Plots of *c* and *k* on Unstable Manifold for Eigenvalues in Table 3A
When $\delta = 0.2$ and $\rho = 0.3$.

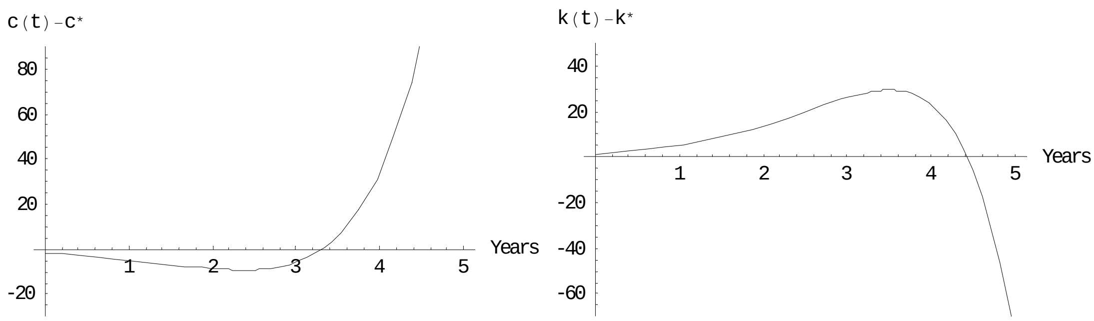
$$A_1 = 0, A_2 = 0, B_1 = 1, B_2 = 0$$



$$A_1 = 0, A_2 = 0, B_1 = \frac{1}{\sqrt{2}}, B_2 = \frac{1}{\sqrt{2}}$$



$$A_1 = 0, A_2 = 0, B_1 = \frac{1}{\sqrt{2}}, B_2 = \frac{-1}{\sqrt{2}}$$



$$A_1 = 0, A_2 = 0, B_1 = 0, B_2 = 1$$