Spurious Regressions in Time Series with Long Memory

Gaowen Wang [1]

[1] Address correspondence to: Gaowen Wang, Department of Banking and Finance, Takming University of Science and Technology, No. 56, Sec. l, Huanshan Rd., Neihu, Taipei 11451, Taiwan. Tel.: 886-2-26585801. E-mail: gwwang@takming.edu.tw.

Abstract

This paper studies the asymptotic distributions of partial sums of fractionally integrated processes which are long memory. The required moment conditions on the innovations of the processes are weak. We also discuss the asymptotic properties of least squares estimators and related test statistics in some spurious regression models that are generated by stationary or non-stationary fractionally integrated processes. We show that even when the fractionally integrated processes are long-range dependent, the asymptotic distributions of the least squares statistics, after appropriately rescaling and normalizing, are functionals of standard Brownian motions rather than of fractional Brownian motions.

JEL Classification numbers: C12, C13, C22.

Keyword: Linear process; Long memory; Martingale differences; Mixing sequence; Spurious regression; Self-normalized sums.

1 Introduction

Suppose that y_t and x_t are generated by the independent random walks

$$y_t = y_{t-1} + v_t, \qquad x_t = x_{t-1} + w_t, \qquad t = 1, 2, \ldots. \tag{1}$$

Consider the ordinary least squares (OLS) regression

$$y_t = \hat{\alpha} + \hat{\beta} x_t + \hat{u}_t, \quad t = 1, \ldots, n. \tag{2}$$

Assuming that v_t and w_t are independent and identically distributed (iid) random variables, Granger and Newbold (1974) showed by simulations that the OLS estimates $\hat{\alpha}$ and $\hat{\beta}$ are frequently found to be significant, that the coefficient of determination R^2 from the regression is often high and that the Durbin-Watson (DW) statistic is close to zero. This is one of the situations in which we have a *nonsense* or *spurious regression*. Phillips (1986) extended (1) to the case that both v_t and w_t are strong mixing random processes, and provided a very elegant asymptotic theory that gives a better understanding of the simulation results. Phillips' results were further extended by Marmol (1995, 1996) to integrated processes of higher integer orders. Up to now, it has been known that the phenomenon of spurious regression also occurs in a wider class of stochastic processes, such as random walk processes with drifts (Molinas, 1986; Entorf, 1997), some particular types of stationary processes (Tsay and Chung, 2000; Granger et al., 2001; Kim et al., 2004) and nonstationary fractionally integrated processes (Cappuccio and Lubian, 1997; Marmol, 1998; Marmol and Reboredo, 1999; Tsay and Chung, 2000).

The fractionally integrated processes were introduced by Granger and Joyeux (1980) and Hosking (1981). Since then, they have become increasingly popular in recent years due to their considerable empirical success in macroeconomics and finance (see, for examples, Robinson, 1994; Baillie, 1996; Henry and Zaffaroni, 2003). For some recent contributions, we refer to excellent books of Doukhan et al. (2003) and Robinson (2003). The main aim of the present paper is to deal with the problem of spurious regression under situations where the underlying processes are driven by nonstationary or stationary fractionally integrated processes. The problem has been extensively studied in Cappuccio and Lubian (1997) and Tsay and Chung (2000). Instead of using the functional central limit theorem for mixing processes used by Phillips (1986), Cappuccio and Lubian (1997) and Tsay and Chung (2000) applied the invariance principle established by Davydov (1970, Theorem 2) to derive the asymptotic distributions of various OLS statistics in the spurious regression models involving fractionally integrated processes. In contrast with the three papers just cited, in this paper we use the invariance principle of Csörgő et al. (2003), i.e., a self-normalized version of weak invariance principle, to re-examine spurious regressions between fractionally integrated processes. The reason we do so is that self-normalization can eliminate or weaken moment assumptions such that the limit theorems for the self-normalized sums and associated test statistics hold under weaker conditions than those required in the classical limit theorems.

In the present paper we are interested in the asymptotic behavior of the regression coefficients and related test statistics from the spurious regression models. The underlying stochastic processes we consider are fractionally integrated processes which are assumed to be long memory (or, say, long-range dependent). In the long memory literature, finite fourth moment conditions on the innovations in the underlying processes are generally required. Such conditions, however, are really restrictive to model either financial or some macroeconomic variables. Instead of using this rather strict assumption, we assume that the innovations can be either iid, martingale-difference or mixing sequence and that their second moments are finite. This extends and generalizes previous work by Cappuccio and Lubian (1997) and Tsay and Chung (2000). We show that the asymptotic distributions of the coefficient estimators and associated test statistics have more interesting structure which appear rather atypical in the sense that they may no longer be functionals of fractional Brownian motions. Instead, the asymptotic distributions we obtain are functionals of standard Brownian motions. From the theoretical point of view, this result in itself is rather interesting and very important.

The paper is organized as follows. Section II presents asymptotic results of self-normalized partial sums of long memory fractionally integrated processes. Applications to the spurious regression problems are made in Section III. In the same section, we make comparisons with related work. The proofs are given in Section IV, and Section V concludes the paper.

Throughout the paper, we use the following notations. $\to_{a.s.}$, $\to_p$, and $\Rightarrow$ denote convergence almost surely, convergence in probability, and weak convergence of probability measures on $D[0,1]$ under the Skorokhod topology, respectively. $O_p(1)$ ($o_p(1)$) stands for a sequence of random variables that is bounded (converges to zero) in probability. For two sequences of real numbers $\{r_n\}$ and $\{s_n\}$, we write $r_n \sim s_n$ if $\lim_{n\to\infty} r_n/s_n = 1$, and $r_n = O(s_n)(o(s_n))$ if the ratio $|r_n/s_n|$ is bounded (converges to zero) for large n. The indicator of a set A is denoted by $\mathbb{1}_{(A)}$. Symbol $=:$ means equality by definition, and $[x]$ denotes the largest integer less than or equal to x.

2 Long memory fractionally integrated processes

Let $\{X_t\}_{t\in\mathbb{Z}}$ be a sequence of iid random variables with zero mean. Let $S_{X,n} = \sum_{t=1}^n X_t$ and $V_{X,n}^2 = \sum_{t=1}^n X_t^2$, $n \in \mathbb{N}$, then the quotient $S_{X,n}/V_{X,n}$ is the so-called self-normalized sum. Recently, Csörgő et al. (2003, Theorem 1) proved that

$$\frac{S_{X,[nr]}}{V_{X,n}} \Rightarrow W(r), \quad 0 \le r \le 1, \tag{3}$$

if and only if

$$\lim_{x\to\infty} \frac{x^2 \mathbb{P}(|X_t| > x)}{\mathbb{E}(X_t^2 \mathbb{1}_{(|X_t|\le x)})} = 0 \tag{4}$$

or, equivalently, if and only if $\mathbb{E}(X_t^2 \mathbb{1}_{(|X_t| \le x)})$ is slowly varying at infinity, where $W(r)$ is a standard Brownian motion. The condition in (4) is equivalent to saying that the distribution of X_t lies in the domain of attraction of the normal law. This is the case whenever X_t have finite variance. For further definitions and details, see Gnedenko and Kolmogorov (1968, p.172), Feller (1971, p.578) and Araujo and Giné (1980, Theorem 6.17).

Moreover, the asymptotic result (3) also holds in the following two important cases. First, let $\{S_{X,t}, \mathscr{F}_t\}$ be a square-integrable martingale whose differences X_t form a stationary ergodic sequence with zero mean and finite variance σ_X^2, where $\mathscr{F}_t$ is an increasing sequence of σ-field generated by $\{X_s : s \le t\}$. Then, $S_{X,[nr]}/V_{X,n} \Rightarrow W(r)$, $0 \le r \le 1$ (see, Hall and Heyde, 1980, Theorem 4.1, p.99). Otherwise, let $\{X_t\}_{t\in\mathbb{Z}}$ be a strictly mixing sequence of random variables with zero mean and finite variance σ_X^2. Suppose

$$0 < \lim_{n\to\infty} \frac{\sigma_X^2 n}{\mathbb{E}(S_{X,n}^2)} = \kappa^2 < \infty.$$

Then, under appropriate regularity conditions, it is well known from Davydov (1968) and Peligrad (1982, Section 2) that

$$\frac{S_{X,[nr]}}{\sqrt{\mathbb{E}(S_{X,n}^2)}} \Rightarrow W(r), \quad 0 \le r \le 1.$$

It is also easy to show that $V_{X,n}^2/(\sigma_X^2 n) \to_p 1$. Putting these facts together yields

$$\frac{S_{X,[nr]}/\sqrt{\mathbb{E}(S_{X,n}^2)}}{V_{X,n}/(\sigma_X\sqrt{n})} = \kappa \frac{S_{X,[nr]}}{V_{X,n}} \Rightarrow W(r), \quad 0 \le r \le 1.$$

In what follows, for simplicity, we assume $\sigma_X^2 < \infty$ and then focus our attention on the long memory properties of fractionally integrated processes.

Let $\{Y_t\}$ be a fractionally integrated process of order d given by $(1-L)^d Y_t = X_t$, $d \in (-0.5, 0.5)$, where L is the backshift operator. Then $\{Y_t\}$ is stationary and invertible, and has the representation

$$\begin{aligned} Y_t &= (1-L)^{-d} X_t = \sum_{j=0}^{\infty} \theta_j X_{t-j}, & (5) \\ \theta_j &= \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)} \sim \frac{j^{-(1-d)}}{\Gamma(d)} \quad \text{as } j \to \infty, & (6) \end{aligned}$$

where $\Gamma(\cdot)$ denotes the gamma function. If $0 < d < 0.5$, the coefficients θ_j are positive and square summable but not summable (i.e., $\sum_{j=0}^{\infty} \theta_j^2 < \infty$ but $\sum_{j=0}^{\infty} \theta_j = \infty$) and then we say that the $\{Y_t\}$ process is long memory or long-range dependent. Whereas, if $-0.5 < d \leq 0$, $\sum_{j=0}^{\infty} |\theta_j| < \infty$ and then we say that the $\{Y_t\}$ process is short memory or short-range dependent. See, Hosking (1981, p.169) and Hall (1992, p.118).

Throughout the paper, we focus on the long memory case (i.e., $0 < d < 0.5$), and assume that

$$\mathbb{E}(X_t^2) + \sum_{j=0}^{\infty} \theta_j^2 < \infty. \qquad (7)$$

This condition guarantees that the $\{Y_t\}$ process defined in (5)–(6) is well-defined (see, Doukhan, 2003, p.47).

Let $S_{Y,n} = \sum_{t=1}^{n} Y_t$ and $V_{Y,n}^2 = \sum_{t=1}^{n} Y_t^2$. Define $\Theta_{s,j} = \sum_{k=1}^{j} \theta_{(s+1)n+1-k}$ for $1 \leq j \leq n$ and $s \geq 0$, and $\Upsilon_d^2 = \sum_{s=0}^{\infty} [(s+1)^d - s^d]^2$. Note that $\Theta_{s,n} \uparrow \infty$ as $n \to \infty$ for every $s \geq 0$ and that $\Upsilon_d^2 < \infty$ (see Lemma 1 below). We are now ready to state the first result, which is useful in deriving the asymptotic properties of some spurious regression models, discussed in the next section.

Theorem 1. *Let $\{Y_t\}$ satisfy (5) and (6) with $d \in (0, 0.5)$, and let $\{X_t\}$ be a sequence of iid random variables with zero mean. Suppose that the condition in (7) holds. Then, as $n \to \infty$,*

(a)

$$\frac{S_{Y,[nr]}}{k_{d,n} V_{X,n}} \Rightarrow W(r), \quad 0 \leq r \leq 1,$$

where $k_{d,n} = \Theta_{0,n} \Upsilon_d$;

(b)

$$\frac{V_{Y,n}^2}{V_{X,n}^2} \to_{a.s.} \sum_{j=0}^{\infty} \theta_j^2;$$

(c)

$$\frac{(\sum_{j=0}^{\infty} \theta_j^2)^{1/2}}{k_{d,n}} \frac{S_{Y,[nr]}}{V_{Y,n}} \Rightarrow W(r), \quad 0 \leq r \leq 1.$$

Theorem 1(a) indicates that the asymptotic distribution of the partial sums of long memory frationally integrated processes $S_{Y,[nr]}$, after normalizing by random variables $k_{d,n} V_{X,n}$, is a standard Brownian motion, which differs sharply from the well-known fractional Brownian motions proposed by previous studies in the long memory literature. We will further discuss this point in the next section. In addition, as shown in Theorem 1(c), the self-normalized partial sums of long memory processes $S_{Y,[nr]}/V_{Y,n}$, after appropriately rescaling, also converge weakly as $n \to \infty$ to the standard Brownian motion.

It is remarkable here that Theorem 1 holds true even when $\{X_t\}_{t\in\mathbb{Z}}$ is either a stationary ergodic martingale difference sequence or a strictly stationary mixing sequence (if we assume, without loss of generality, that $\kappa = 1$); see the discussion above in this section. Moreover, our moment conditions are weaker than those in Davydov (1970), Cappuccio and Lubian (1997) and Tsay and Chung (2000).

In the next section, we will use Theorem 1 to study the asymptotic behavior of OLS estimators and associated test statistics in some spurious regression models.

3 Spurious regressions with long memory

Let $\{a_t\}_{t\in\mathbb{Z}}$ and $\{b_t\}_{t\in\mathbb{Z}}$ be two sequences of iid random variables with zero means and finite variances σ_a^2 and σ_b^2. Assume that y_t and x_t are mutually independent for all t and generated, respectively, from the following data generating processes:

$$y_t = y_{t-1} + v_t, \qquad (1-L)^{d_1} v_t = a_t, \qquad d_1 \in (0, 0.5), \tag{8}$$

$$x_t = x_{t-1} + w_t, \qquad (1-L)^{d_2} w_t = b_t, \qquad d_2 \in (0, 0.5). \tag{9}$$

Similar to (5) and (6), the $\{v_t\}$ and $\{w_t\}$ processes are stationary and invertible, and have the representations

$$v_t = \sum_{j=0}^{\infty} \theta_{a,j} a_{t-j}, \quad \theta_{a,j} = \frac{\Gamma(j+d_1)}{\Gamma(j+1)\Gamma(d_1)} \sim \frac{j^{-(1-d_1)}}{\Gamma(d_1)} \quad \text{as } j \to \infty, \tag{10}$$

$$w_t = \sum_{j=0}^{\infty} \theta_{b,j} b_{t-j}, \quad \theta_{b,j} = \frac{\Gamma(j+d_2)}{\Gamma(j+1)\Gamma(d_2)} \sim \frac{j^{-(1-d_2)}}{\Gamma(d_2)} \quad \text{as } j \to \infty. \tag{11}$$

Similar to (7), assume the following conditions hold:

$$\mathbb{E}(a_t^2) + \sum_{j=0}^{\infty} \theta_{a,j}^2 < \infty \quad \text{and} \quad \mathbb{E}(b_t^2) + \sum_{j=0}^{\infty} \theta_{b,j}^2 < \infty. \tag{12}$$

Note also that if $d_1, d_2 \in (0, 0.5)$, then $\sum_{j=0}^{\infty} \theta_{a,j} = \infty$ and $\sum_{j=0}^{\infty} \theta_{b,j} = \infty$.

Let $S_{v,n} = \sum_{t=1}^{n} v_t$, $V_{a,n}^2 = \sum_{t=1}^{n} a_t^2$, $S_{w,n} = \sum_{t=1}^{n} w_t$, and $V_{b,n}^2 = \sum_{t=1}^{n} b_t^2$. Define $\Theta_{a,s,j} = \sum_{k=1}^{j} \theta_{a,(s+1)n+1-k}$ and $\Theta_{b,s,j} = \sum_{k=1}^{j} \theta_{b,(s+1)n+1-k}$ for $1 \leq j \leq n$ and $s \geq 0$. Also define $\Upsilon_{d_1}^2 = \sum_{s=0}^{\infty} [(s+1)^{d_1} - s^{d_1}]^2$ and $\Upsilon_{d_2}^2 = \sum_{s=0}^{\infty} [(s+1)^{d_2} - s^{d_2}]^2$. Now let $k_{d_1,n} = \Theta_{a,0,n} \Upsilon_{d_1}$ and $k_{d_2,n} = \Theta_{b,0,n} \Upsilon_{d_2}$. Then by Theorem 1(a), we immediately have that as $n \to \infty$,

$$\frac{S_{v,[nr]}}{k_{d_1,n} V_{a,n}} \Rightarrow W_a(r), \quad \frac{S_{w,[nr]}}{k_{d_2,n} V_{b,n}} \Rightarrow W_b(r), \quad 0 \leq r \leq 1, \tag{13}$$

where $W_a(r)$ and $W_b(r)$ are two independent standard Brownian motions.

We now consider the OLS regression (2). Throughout the paper, we denote by $\hat{s}^2$, R^2, DW, t_α and t_β the estimated variance of the OLS residuals $\hat{u}_t$, the coefficient of determination, the Durbin-Watson statistic and the conventional t-ratios for the intercept estimator $\hat{\alpha}$ and the slope estimator $\hat{\beta}$. These notations will be used repeatedly in the rest of the paper without further reference.

Theorem 2. *Suppose that y_t and x_t are generated by (8) and (9) with coefficients $\theta_{a,j}$ and $\theta_{b,j}$ satisfying (10) and (11), respectively. Assume that the conditions in (12) hold. If the regression (2) is estimated by OLS, then as $n \to \infty$,*

(a)

$$\frac{k_{d_2,n}V_{b,n}}{k_{d_1,n}V_{a,n}}\hat{\beta} \Rightarrow \frac{\int_0^1 W_a(r)W_b(r)\mathrm{d}r - \left[\int_0^1 W_a(r)\mathrm{d}r\right]\left[\int_0^1 W_b(r)\mathrm{d}r\right]}{\int_0^1 W_b^2(r)\mathrm{d}r - \left[\int_0^1 W_b(r)\mathrm{d}r\right]^2} =: \xi_{T2,\beta};$$

(b)

$$\frac{\hat{\alpha}}{k_{d_1,n}V_{a,n}} \Rightarrow \int_0^1 W_a(r)\mathrm{d}r - \xi_{T2,\beta}\int_0^1 W_b(r)\mathrm{d}r =: \xi_{T2,\alpha};$$

(c)

$$\begin{aligned}\frac{\hat{s}^2}{k_{d_1,n}^2 V_{a,n}^2} \quad \Rightarrow \quad & \int_0^1 W_a^2(r)\mathrm{d}r - \left[\int_0^1 W_a(r)\mathrm{d}r\right]^2 \\ & -\xi_{T2,\beta}^2\left\{\int_0^1 W_b^2(r)\mathrm{d}r - \left[\int_0^1 W_b(r)\mathrm{d}r\right]^2\right\} =: \xi_{T2,s^2};\end{aligned}$$

(d)

$$\frac{t_\beta}{\sqrt{n}} \Rightarrow \frac{\xi_{T2,\beta}}{\xi_{T2,s^2}^{1/2}}\left\{\int_0^1 W_b^2(r)\mathrm{d}r - \left[\int_0^1 W_b(r)\mathrm{d}r\right]^2\right\}^{1/2};$$

(e)

$$\frac{t_\alpha}{\sqrt{n}} \Rightarrow \frac{\xi_{T2,\alpha}}{\xi_{T2,s^2}^{1/2}}\left\{\frac{\int_0^1 W_b^2(r)\mathrm{d}r - \left[\int_0^1 W_b(r)\mathrm{d}r\right]^2}{\int_0^1 W_b^2(r)\mathrm{d}r}\right\}^{1/2};$$

(f)

$$R^2 \Rightarrow \xi_{T2,\beta}^2\,\frac{\int_0^1 W_b^2(r)\mathrm{d}r - \left[\int_0^1 W_b(r)\mathrm{d}r\right]^2}{\int_0^1 W_a^2(r)\mathrm{d}r - \left[\int_0^1 W_a(r)\mathrm{d}r\right]^2};$$

(g)

$$DW \to_p 0.$$

Theorem 2 shows three important features. First, as remarked above, Theorem 2 holds true even when a_t and b_t, $t \in \mathbb{Z}$, are either stationary ergodic martingale difference sequences or strictly stationary mixing sequences (if $\kappa = 1$). Second, it shows that the asymptotic distributions of the OLS estimators and related test statistics from the spurious regression model are functionals of standard Brownian motions rather than the ones of fractional Brownian motions. As a consequence, our results are closer to those in Phillips (1986) than the ones in Cappuccio and Lubian (1997) and Tsay and Chung (2000). Third, the orders of $\hat{\beta}$, $\hat{\alpha}$, $\hat{s}^2$, t_β, t_α, R^2 and DW are identical to what they are in Tsay and Chung (2000, Theorem 1), but, again, the asymptotic distributions are different and the required moment conditions are weak.

The following corollary is a special case of Theorem 2 and is quite similar to the case studied in Cappuccio and Lubian (1997).

Corollary 1. *Under the same conditions as in Theorem 2, suppose in addition that $d_1 = d_2 = d$. If the regression (2) is estimated by OLS, then, as $n \to \infty$, the asymptotic distributions of $\hat{\beta}$, $t_\beta/\sqrt{n}$, $t_\alpha/\sqrt{n}$, R^2 and DW are the same as the corresponding ones in Phillips (1986, Theorem 1).*

Again, the sharp differences between the results of Corollary 1 and of

Theorem 2.3 in Cappuccio and Lubian (1997) are similar to those described below Theorem 2.

We now consider two spurious regression models as follows:

$$y_t = \hat{\alpha} + \hat{\beta} w_t + \hat{u}_t, \tag{14}$$

$$v_t = \hat{\alpha} + \hat{\beta} x_t + \hat{u}_t. \tag{15}$$

The asymptotics of these two regression models have been studied in Theorem 3 and Theorem 4 of Tsay and Chung (2000), respectively. A comparison with their work is given in the following two corollaries.

Corollary 2. *Suppose that y_t and w_t are generated by (8) and (9) with coefficients $\theta_{a,j}$ and $\theta_{b,j}$ satisfying (10) and (11), respectively. Assume that the conditions in (12) hold. If the regression (14) is estimated by OLS, then, as $n \to \infty$,*

(a)

$$\frac{V_{b,n} \sum_{j=0}^{\infty} \theta_{b,j}^2}{V_{a,n} k_{d_1,n} k_{d_2,n}} \hat{\beta} \Rightarrow \int_0^1 W_a(r) \mathrm{d}W_b(r) - W_b(1) \int_0^1 W_a(r) \mathrm{d}r =: \xi_{C2,\beta};$$

(b)

$$\frac{\hat{\alpha}}{k_{d_1,n} V_{a,n}} \Rightarrow \int_0^1 W_a(r) \mathrm{d}r =: \xi_{C2,\alpha};$$

(c)

$$\frac{\hat{s}^2}{k_{d_1,n}^2 V_{a,n}^2} \Rightarrow \int_0^1 W_a^2(r) \mathrm{d}r - \left[\int_0^1 W_a(r) \mathrm{d}r \right]^2 =: \xi_{C2,s^2};$$

(d)

$$\frac{\left(\sum_{j=0}^{\infty} \theta_{b,j}^2 \right)^{1/2}}{k_{d_2,n}} t_\beta \Rightarrow \frac{\xi_{C2,\beta}}{\xi_{C2,s^2}^{1/2}};$$

(e)

$$\frac{t_\alpha}{\sqrt{n}} \Rightarrow \frac{\xi_{C2,\alpha}}{\xi_{C2,s^2}^{1/2}};$$

(f)

$$\frac{n\sum_{j=0}^{\infty}\theta_{b,j}^2}{k_{d_2,n}^2}R^2 \Rightarrow \frac{\xi_{C2,\beta}^2}{\int_0^1 W_a^2(r)\mathrm{d}r - \left[\int_0^1 W_a(r)\mathrm{d}r\right]^2};$$

(g)

$$DW \to_p 0.$$

Corollary 3. *Suppose that v_t and x_t are generated by (8) and (9) with coefficients $\theta_{a,j}$ and $\theta_{b,j}$ satisfying (10) and (11), respectively. Assume that the conditions in (12) hold. If the regression (15) is estimated by OLS, then, as $n \to \infty$,*

(a)

$$\frac{nk_{d_2,n}V_{b,n}}{k_{d_1,n}V_{a,n}}\hat{\beta} \Rightarrow \frac{\int_0^1 W_b(r)\mathrm{d}W_a(r) - W_a(1)\int_0^1 W_b(r)\mathrm{d}r}{\int_0^1 W_b^2(r)\mathrm{d}r - \left[\int_0^1 W_b(r)\mathrm{d}r\right]^2} =: \xi_{C3,\beta};$$

(b)

$$\frac{n}{k_{d_1,n}V_{a,n}}\hat{\alpha} \Rightarrow W_a(1) - \xi_{C3,\beta}\int_0^1 W_b(r)\mathrm{d}r =: \xi_{C3,\alpha};$$

(c)

$$\hat{s}^2 \to_p \sigma_a^2\sum_{j=0}^{\infty}\theta_{a,j}^2;$$

(d)

$$\frac{(\sum_{j=0}^{\infty}\theta_{a,j}^2)^{1/2}}{k_{d_1,n}}t_\beta \Rightarrow \xi_{C3,\beta}\left\{\int_0^1 W_b^2(r)\mathrm{d}r - \left[\int_0^1 W_b(r)\mathrm{d}r\right]^2\right\}^{1/2};$$

(e)

$$\frac{(\sum_{j=0}^{\infty}\theta_{a,j}^2)^{1/2}}{k_{d_1,n}} t_\alpha \Rightarrow \xi_{C3,\alpha}\left\{\frac{\int_0^1 W_b^2(r)\mathrm{d}r - \left[\int_0^1 W_b(r)\mathrm{d}r\right]^2}{\int_0^1 W_b^2(r)\mathrm{d}r}\right\}^{1/2};$$

(f)

$$\frac{n\sum_{j=0}^{\infty}\theta_{a,j}^2}{k_{d_1,n}^2} R^2 \Rightarrow \xi_{C3,\beta}^2\left\{\int_0^1 W_b^2(r)\mathrm{d}r - \left[\int_0^1 W_b(r)\mathrm{d}r\right]^2\right\};$$

(g)

$$DW \to_p 2\left[1 - \frac{\sum_{j=0}^{\infty}\theta_{a,j}\theta_{a,j+1}}{\sum_{j=0}^{\infty}\theta_{a,j}^2}\right].$$

Though the orders of the OLS estimators and associated test statistics in Corollary 2 and Corollary 3 are the same as the corresponding ones in Theorem 3 and Theorem 4 of Tsay and Chung (2000), respectively, however, the asymptotic distributions are different and derived under slightly weaker moment conditions, as mentioned above. Furthermore, it may be noteworthy that, after appropriately rescaling and normalizing, the asymptotic distributions of $\hat{\beta}$, t_β and R^2 in Corollary 2 and of $\hat{\beta}$, $\hat{\alpha}$, t_β, t_α and R^2 in Corollary 3 are available in this paper, whereas they are not provided in the paper by Tsay and Chung (2000). We remark again that the results in Corollaries 2–3 are also asymptotically valid even when a_t and b_t, $t \in \mathbb{Z}$, are either stationary ergodic martingale difference sequences or strictly stationary mixing sequences (if $\kappa = 1$).

The following corollary presents a special case of Corollary 3 and is of independent interest.

Corollary 4. *Suppose that v_t and x_t are generated by (8) and (9) with coefficients $\theta_{a,j}$ and $\theta_{b,j}$ satisfying (10) and (11), respectively. Assume that the conditions in (12) hold. Consider a spurious regression of the form $v_t = \hat{\beta} x_t + \hat{u}_t$, which is estimated by OLS. Then, as $n \to \infty$,*

$$\frac{n k_{d_2,n} V_{b,n}}{k_{d_1,n} V_{a,n}} \hat{\beta} \Rightarrow \frac{\int_0^1 W_b(r) \mathrm{d} W_a(r)}{\int_0^1 W_b^2(r) \mathrm{d} r} \sim \mathcal{N}(0,1).$$

Note here that if $d_1 = d_2$, then $k_{d_1,n} = k_{d_2,n}$, and that if, without loss of generality, $\mathbb{E}(a_t^2) = \mathbb{E}(b_t^2)$, then the law of large numbers gives $V_{a,n}^2 / V_{b,n}^2 \to_p 1$ as $n \to \infty$. As a result, we have $n\hat{\beta} \Rightarrow \mathcal{N}(0,1)$. This is an interesting case, since the statistic has an asymptotic standard normal distribution and is nuisance parameter free.

4 Proofs

In this section, we will prove Theorems 1–2 and Corollaries 1–4. We start with the following notation and lemmas. Let $S_{j,X,n} = \sum_{t=1}^n X_{t-j}$ and $V_{j,X,n}^2 = \sum_{t=1}^n X_{t-j}^2$, $j \geq 0$. Further, for simplicity, write $S_{X,n} = S_{0,X,n}$ and $V_{X,n}^2 = V_{0,X,n}^2$.

Lemma 1. *If the coefficients θ_j are of the form as in (6) with $d \in (0, 0.5)$, then as $n \to \infty$,*

$$\begin{aligned} \sum_{s=0}^{\infty} (\Theta_{s,n}/\Theta_{0,n})^2 &\sim \sum_{s=0}^{\infty} \left[(s+1)^d - s^d \right]^2 \\ &\sim \frac{-1}{2d+1} + \frac{[\Gamma(d+1)]^2}{\Gamma(2d+2)\sin[\pi(d+1/2)]} < \infty. \quad (16) \end{aligned}$$

Proof of Lemma 1. Note that $\theta_0 = 1$ and assume, without loss of generality, that $\theta_j = j^{-(1-d)}/\Gamma(d)$ for $j \geq 1$. Recall that $\Theta_{s,n} = \sum_{k=1}^{n} \theta_{(s+1)n+1-k}$, $s \geq 0$. Then, standard calculus yields that as $n \to \infty$,

$$\begin{aligned}
\Theta_{s,n} &= \frac{1}{\Gamma(d)} \left[\frac{1}{(sn+1)^{1-d}} + \ldots + \frac{1}{(sn+n)^{1-d}} \right] \\
&\sim \frac{n^d}{\Gamma(d)} \int_0^1 \frac{1}{(s+x)^{1-d}} \mathrm{d}x \\
&= \frac{n^d}{\Gamma(d)d} \left[(s+1)^d - s^d \right]. \qquad (17)
\end{aligned}$$

It implies that $\Theta_{0,n} \sim n^d/[\Gamma(d)d]$ and thus that $\sum_{s=0}^{\infty} [\Theta_{s,n}/\Theta_{0,n}]^2 \sim \sum_{s=0}^{\infty} [(s+1)^d - s^d]^2$. Since $\sum_{s=0}^{\infty} [(s+1)^d - s^d]^2 \sim \int_0^{\infty} [(s+1)^d - s^d]^2 \mathrm{d}s$, result (16) comes directly from equation (9.3) in Taqqu (2003, p.28) by noting that the Hurst parameter $H = d + 1/2$. This completes the proof. $\square$

Proof of Theorem 1. For part (a), recall that $V_{j,X,n}^2 = \sum_{t=1}^{n} X_{t-j}^2$ for $j \geq 0$. Write

$$\begin{aligned}
\frac{S_{Y,[nr]}}{\Theta_{0,n} V_{X,n}} &= \frac{\sum_{j=0}^{n} \theta_j \sum_{t=1}^{[nr]} X_{t-j} + \sum_{j=n+1}^{2n} \theta_j \sum_{t=1}^{[nr]} X_{t-j} + \ldots}{\Theta_{0,n} V_{X,n}} \\
&= \frac{\sum_{j=0}^{n} \theta_j \sum_{t=1}^{[nr]} X_t + \sum_{j=0}^{n} \theta_j \left(\sum_{t=1}^{[nr]} X_{t-j} - \sum_{t=1}^{[nr]} X_t \right)}{\Theta_{0,n} V_{X,n}} \\
&\quad + \frac{\sum_{j=n+1}^{2n} \theta_j \sum_{t=1}^{[nr]} X_{-n+t} + \sum_{j=n+1}^{2n} \theta_j \left(\sum_{t=1}^{[nr]} X_{t-j} - \sum_{t=1}^{[nr]} X_{-n+t} \right)}{\Theta_{0,n} V_{X,n}} + \ldots \\
&= \frac{\theta_0}{\Theta_{0,n}} \frac{\sum_{t=1}^{[nr]} X_t}{V_{X,n}} + \sum_{s=0}^{\infty} \left\{ \frac{\Theta_{s,n}}{\Theta_{0,n}} \frac{V_{sn,X,n}}{V_{X,n}} \left[\frac{\sum_{t=1}^{[nr]} X_{-sn+t}}{V_{sn,X,n}} \right.\right. \\
&\qquad \left.\left. + \sum_{j=sn+1}^{sn+n} \frac{\theta_j}{\Theta_{s,n}} \frac{\sum_{t=1}^{[nr]} X_{t-j} - \sum_{t=1}^{[nr]} X_{-sn+t}}{V_{sn,X,n}} \right] \right\}. \qquad (18)
\end{aligned}$$

Since $\theta_0 = 1$ and $\Theta_{0,n} \sim n^d/[\Gamma(d)d] \uparrow \infty$ as $n \to \infty$, the first term is $o_p(1)$

by (3). Noting that for $j \geq 1$,

$$S_{j,X,[nr]} - S_{j-1,X,[nr]} = \sum_{t=1}^{[nr]} X_{t-j} - \sum_{t=1}^{[nr]} X_{t-(j-1)} = X_{1-j} - X_{[nr]-(j-1)}, \quad (19)$$

by a straightforward calculation, the second term inside the square brackets on the right hand side of (18) can be written as

$$\begin{aligned}
& \sum_{j=sn+1}^{sn+n} \frac{\theta_j}{\Theta_{s,n}} \frac{\sum_{t=1}^{[nr]} X_{t-j} - \sum_{t=1}^{[nr]} X_{-sn+t}}{V_{sn,X,n}} \\
= & \sum_{j=sn+1}^{sn+n} \frac{\theta_j}{\Theta_{s,n}} \frac{S_{j,X,[nr]} - S_{sn,X,[nr]}}{V_{sn,X,n}} \\
= & \sum_{j=sn+1}^{sn+n} \frac{\theta_j}{\Theta_{s,n} V_{sn,X,n}} \left\{ [S_{j,X,[nr]} - S_{j-1,X,[nr]}] + [S_{j-1,X,[nr]} - S_{j-2,X,[nr]}] \right. \\
& \qquad \left. + \ldots + [S_{sn+1,X,[nr]} - S_{sn,X,[nr]}] \right\} \\
= & \sum_{j=sn+1}^{sn+n} \frac{\theta_j}{\Theta_{s,n} V_{sn,X,n}} \left\{ [X_{1-j} - X_{[nr]-(j-1)}] + [X_{1-(j-1)} - X_{[nr]-(j-2)}] \right. \\
& \qquad \left. + \ldots + [X_{1-(sn+1)} - X_{[nr]-sn}] \right\} \\
= & \frac{1}{\Theta_{s,n} V_{sn,X,n}} \left\{ (\theta_{sn+1} + \ldots + \theta_{sn+n})(X_{-sn} - X_{[nr]-sn}) \right. \\
& \quad + (\theta_{sn+2} + \ldots + \theta_{sn+n})(X_{-sn-1} - X_{[nr]-sn-1}) + \cdots \\
& \quad \left. + \theta_{sn+n}(X_{-sn-(n-1)} - X_{[nr]-sn-(n-1)}) \right\} \\
= & \sum_{j=1}^{n} \frac{\Theta_{s,j}}{\Theta_{s,n}} \frac{X_{-sn-n+j} - X_{[nr]-sn-n+j}}{V_{sn,X,n}}. \qquad (20)
\end{aligned}$$

Noting that $V_{i,X,n}/V_{j,X,n} \rightarrow_{a.s.} 1$ for $i \neq j$, it then follows from (3) that

$$\begin{aligned}
\sum_{j=1}^{n} \frac{X_{-sn-n+j} - X_{[nr]-sn-n+j}}{V_{sn,X,n}} &= \frac{S_{sn+n,X,n} - S_{sn+n-[nr],X,n}}{V_{sn,X,n}} \\
= \frac{S_{sn+n,X,n}}{V_{sn+n,X,n}} \frac{V_{sn+n,X,n}}{V_{sn,X,n}} &- \frac{S_{sn+n-[nr],X,n}}{V_{sn+n-[nr],X,n}} \frac{V_{sn+n-[nr],X,n}}{V_{sn,X,n}} \quad (21)
\end{aligned}$$

is stochastically bounded. This fact, together with $\Theta_{s,n} \uparrow \infty$ as $n \rightarrow \infty$ for every $s \geq 0$, implies that

$$\sum_{j=1}^{n} \frac{\Theta_{s,j}}{\Theta_{s,n}} \frac{X_{-sn-n+j} - X_{[nr]-sn-n+j}}{V_{sn,X,n}} \rightarrow_{a.s.} 0 \tag{22}$$

by Kronecker's lemma (see Petrov, 1995, p.209). As for the first term inside the square brackets on the right hand side of (18), similar to (3), we have $\sum_{t=1}^{[nr]} X_{-sn+t}/V_{sn,X,n} \Rightarrow W_s(r)$, where $0 \leq r \leq 1$ and $s \geq 0$. As a result, the second term in the square brackets is dominated by the first term.

In fact, $W_0(r)$ is identical to the standard Brownian motion $W(r)$ given in (3) and $W_s(r)$, $s \geq 1$, can be seen to be independent copies of $W(r)$. Thus, for fixed r, they are normally distributed random variables with mean zero and variance r. Note again that $V_{sn,X,n}/V_{X,n} \rightarrow_{a.s.} 1$ for $s \geq 1$, and from Lemma 1 that $\Theta_{s,n}/\Theta_{0,n} \sim [(s+1)^d - s^d]$. Recall that $k_{d,n} = \Theta_{0,n}\Upsilon_d$, where $\Upsilon_d^2 = \sum_{s=0}^{\infty}[(s+1)^d - s^d]^2$. Putting these results together, it then follows from the reproductive property of the normal distribution, Lemma 1 and (18) that as $n \rightarrow \infty$,

$$\begin{aligned} \frac{S_{Y,[nr]}}{k_{d,n} V_{X,n}} &= \frac{\sum_{s=0}^{\infty}[(s+1)^d - s^d] \sum_{t=1}^{[nr]} X_{-sn+t}/V_{sn,X,n}}{\Upsilon_d} + o_p(1) \\ &\Rightarrow \frac{\sum_{s=0}^{\infty}[(s+1)^d - s^d] W_s(r)}{\Upsilon_d} \\ &\stackrel{d}{=} W(r) \sim \mathcal{N}(0, r), \end{aligned} \tag{23}$$

where $\stackrel{d}{=}$ denotes equality in distribution. This completes the proof of part (a).

For part (b), because X_t are stationary ergodic (Stout, 1974, Lemma 3.5.8, p.182), it follows from Theorem 3.5.8 of Stout (1974, p.182) that the

$\{Y_t\}$ process is also stationary ergodic with

$$\mathbb{E}(Y_t^2) = \sum_{j=0}^{\infty} \theta_j^2 \sigma_X^2. \tag{24}$$

By using the stationary ergodic theorem (Stout, 1974, Theorem 3.5.7) and (24), we have that

$$\frac{V_{Y,n}^2}{V_{X,n}^2} = \frac{V_{Y,n}^2/n}{V_{X,n}^2/n} \to_{a.s.} \sum_{j=0}^{\infty} \theta_j^2 \quad \text{as} \quad n \to \infty. \tag{25}$$

This completes the proof of part (b).

For part (c), dividing part (a) by the square root of part (b) and then multiplying by $(\sum_{j=0}^{\infty} \theta_j^2)^{1/2}$ yields the desired result. □

Now assume, without loss of generality, that $y_t = x_t = 0$ for all $t \leq 0$. It implies that $y_t = \sum_{k=1}^{t} v_k = S_{v,t}$ and $x_t = \sum_{k=1}^{t} w_k = S_{w,t}$ for all $t > 0$. It then follows from (13) that for $(t-1)/n \leq r < t/n$, $t = 1, \ldots, n$,

$$\frac{y_{t-1}}{k_{d_1,n} V_{a,n}} = \frac{S_{v,t-1}}{k_{d_1,n} V_{a,n}} \Rightarrow W_a(r), \qquad \frac{x_{t-1}}{k_{d_2,n} V_{b,n}} = \frac{S_{w,t-1}}{k_{d_2,n} V_{b,n}} \Rightarrow W_b(r). \tag{26}$$

The key for proving Theorem 2 and our other results is the following lemma.

Lemma 2. *Let y_t and x_t be generated as in (8) and (9), where v_t and w_t have the representations given in (10) and (11), respectively. Suppose that the conditions in (12) hold. Then, as $n \to \infty$,*

(a)

$$\frac{1}{n k_{d_1,n} V_{a,n}} \sum_{t=1}^{n} y_t \Rightarrow \int_0^1 W_a(r) \mathrm{d}r,$$

$$\frac{1}{n k_{d_2,n} V_{b,n}} \sum_{t=1}^{n} x_t \Rightarrow \int_0^1 W_b(r) \mathrm{d}r;$$

(b)

$$\frac{1}{n}\sum_{t=1}^{n} v_t w_t \to_p 0;$$

(c)

$$\frac{1}{k_{d_2,n}V_{b,n}k_{d_1,n}V_{a,n}}\sum_{t=1}^{n} x_{t-1}v_t \Rightarrow \int_0^1 W_b(r)\mathrm{d}W_a(r),$$

$$\frac{1}{k_{d_2,n}^2 V_{b,n}^2}\sum_{t=1}^{n} x_{t-1}w_t \Rightarrow \int_0^1 W_b(r)\mathrm{d}W_b(r),$$

$$\frac{1}{k_{d_1,n}V_{a,n}k_{d_2,n}V_{b,n}}\sum_{t=1}^{n} y_{t-1}w_t \Rightarrow \int_0^1 W_a(r)\mathrm{d}W_b(r),$$

$$\frac{1}{k_{d_1,n}^2 V_{a,n}^2}\sum_{t=1}^{n} y_{t-1}v_t \Rightarrow \int_0^1 W_a(r)\mathrm{d}W_a(r);$$

(d)

$$\frac{1}{nk_{d_2,n}^2 V_{b,n}^2}\sum_{t=1}^{n} x_t^2 \Rightarrow \int_0^1 W_b^2(r)\mathrm{d}r,$$

$$\frac{1}{nk_{d_1,n}^2 V_{a,n}^2}\sum_{t=1}^{n} y_t^2 \Rightarrow \int_0^1 W_a^2(r)\mathrm{d}r;$$

(e)

$$\frac{1}{nk_{d_1,n}V_{a,n}k_{d_2,n}V_{b,n}}\sum_{t=1}^{n} y_t x_t \Rightarrow \int_0^1 W_a(r)W_b(r)\mathrm{d}r.$$

Proof of Lemma 2. For part (a), it follows from (26) and the continuous mapping theorem that as $n \to \infty$,

$$\begin{aligned}\frac{1}{nk_{d_1,n}V_{a,n}}\sum_{t=1}^{n} y_t &= \frac{1}{nk_{d_1,n}V_{a,n}}\sum_{t=1}^{n}(y_{t-1}+v_t) = \frac{1}{n}\sum_{t=1}^{n}\frac{y_{t-1}}{k_{d_1,n}V_{a,n}} + o_p(1)\\ &\Rightarrow \int_0^1 W_a(r)\mathrm{d}r. \end{aligned} \tag{27}$$

The result for $(nk_{d_2,n}V_{b,n})^{-1}\sum_{t=1}^{n} x_t$ can be proved in a similar way.

For part (b), it follows from Hall (1992, p.118) and Tsay and Chung (2000, p.176) that

$$\operatorname{var}\left(\sum_{t=1}^{n} v_t w_t\right) = \begin{cases} O\left(n^{2d_1+2d_2}\right) & \text{if } d_1 + d_2 > 0.5, \\ O\left(n \ln(n)\right) & \text{if } d_1 + d_2 = 0.5, \\ O\left(n\right) & \text{otherwise.} \end{cases} \tag{28}$$

Since $d_1 + d_2 < 1$, we have that $\operatorname{var}(n^{-1}\sum_{t=1}^{n} v_t w_t) \to 0$ as $n \to \infty$. Hence it follows easily from the weak law of large numbers (e.g., Petrov, 1995, Theorem 4.16) that $n^{-1}\sum_{t=1}^{n} v_t w_t \to_p 0$ as $n \to \infty$.

To prove part (c) we use arguments analogous to those of Phillips (1986, 1987). We first prove the first case of part (c). Define

$$T_{v,n}(r) = \frac{1}{k_{d_1,n}V_{a,n}} S_{v,[nr]} + \frac{nr - [nr]}{k_{d_1,n}V_{a,n}} v_{[nr]+1}, \tag{29}$$

$$T_{w,n}(r) = \frac{1}{k_{d_2,n}V_{b,n}} S_{w,[nr]} + \frac{nr - [nr]}{k_{d_2,n}V_{b,n}} w_{[nr]+1}, \tag{30}$$

$(t-1)/n \leq r < t/n$, $t = 1, \ldots, n$. It then follows that $T_{v,n}(r) \Rightarrow W_a(r)$, $T_{w,n}(r) \Rightarrow W_b(r)$, $\mathrm{d}T_{v,n}(r) = nv_t\mathrm{d}r/(k_{d_1,n}V_{a,n})$, $\mathrm{d}T_{w,n}(r) = nw_t\mathrm{d}r/(k_{d_2,n}V_{b,n})$ and

$$\int_{(t-1)/n}^{t/n} T_{w,n}(r)\mathrm{d}T_{v,n}(r) = \frac{S_{w,t-1}v_t}{k_{d_2,n}V_{b,n}k_{d_1,n}V_{a,n}} + \frac{w_t v_t}{2k_{d_2,n}V_{b,n}k_{d_1,n}V_{a,n}}. \tag{31}$$

Summing (31) over $t = 1, \ldots, n$ and rearranging yields that as $n \to \infty$,

$$\begin{aligned} \frac{\sum_{t=1}^{n} S_{w,t-1}v_t}{k_{d_2,n}V_{b,n}k_{d_1,n}V_{a,n}} &= \sum_{t=1}^{n}\int_{(t-1)/n}^{t/n} T_{w,n}(r)\mathrm{d}T_{v,n}(r) - \sum_{t=1}^{n}\frac{w_t v_t}{2k_{d_2,n}V_{b,n}k_{d_1,n}V_{a,n}} \\ &= \sum_{t=1}^{n}\int_{(t-1)/n}^{t/n} T_{w,n}(r)\mathrm{d}T_{v,n}(r) + o_p(1) \end{aligned}$$

$$\Rightarrow \int_0^1 W_b(r)\mathrm{d}W_a(r).$$

The proofs of the remaining cases are exactly analogous to the proof just given, so we omit them here.

For part (d), note that $\sum_{t=1}^n x_t^2 = \sum_{t=1}^n (x_{t-1}^2 + 2x_{t-1}w_t + w_t^2)$. By Theorem 1(b), (26), part (c) and the continuous mapping theorem, we have that as $n \to \infty$,

$$\frac{1}{nk_{d_2,n}^2 V_{b,n}^2}\sum_{t=1}^n x_t^2 = \frac{1}{nk_{d_2,n}^2 V_{b,n}^2}\sum_{t=1}^n x_{t-1}^2 + o_p(1) \Rightarrow \int_0^1 W_b^2(r)\mathrm{d}r. \tag{32}$$

The result for $(nk_{d_1,n}^2 V_{a,n}^2)^{-1}\sum_{t=1}^n y_t^2$ can be proved in a similar manner.

Note that $\sum_{t=1}^n x_t y_t = \sum_{t=1}^n (x_{t-1}y_{t-1} + x_{t-1}v_t + y_{t-1}w_t + v_t w_t)$. By (26), parts (b)–(c) and the continuous mapping theorem, the proof of part (e) is then similar to that just given for part (d), and therefore is omitted. □

Proof of Theorem 2. For part (a), put $\bar{x} = n^{-1}\sum_{t=1}^n x_t$ and $\bar{y} = n^{-1}\sum_{t=1}^n y_t$. Note that

$$\hat{\beta} = \frac{\sum_{t=1}^n (x_t - \bar{x})(y_t - \bar{y})}{\sum_{t=1}^n (x_t - \bar{x})^2} = \frac{\sum_{t=1}^n x_t y_t - n^{-1}\sum_{t=1}^n x_t \sum_{t=1}^n y_t}{\sum_{t=1}^n x_t^2 - n^{-1}(\sum_{t=1}^n x_t)^2}.$$

Then, as $n \to \infty$,

$$\begin{aligned}
\frac{k_{d_2,n}V_{b,n}}{k_{d_1,n}V_{a,n}}\hat{\beta} &= \frac{\dfrac{\sum_{t=1}^n x_t y_t}{nk_{d_1,n}V_{a,n}k_{d_2,n}V_{b,n}} - \left[\dfrac{\sum_{t=1}^n x_t}{nk_{d_2,n}V_{b,n}}\right]\left[\dfrac{\sum_{t=1}^n y_t}{nk_{d_1,n}V_{a,n}}\right]}{\dfrac{\sum_{t=1}^n x_t^2}{nk_{d_2,n}^2 V_{b,n}^2} - \left(\dfrac{\sum_{t=1}^n x_t}{nk_{d_2,n}V_{b,n}}\right)^2} \\
&\Rightarrow \frac{\int_0^1 W_a(r)W_b(r)\mathrm{d}r - \left[\int_0^1 W_a(r)\mathrm{d}r\right]\left[\int_0^1 W_b(r)\mathrm{d}r\right]}{\int_0^1 W_b^2(r)\mathrm{d}r - \left[\int_0^1 W_b(r)\mathrm{d}r\right]^2} =: \xi_{T2,\beta}
\end{aligned}$$

by parts (a), (d) and (e) of Lemma 2.

To prove part (b) we first note that $\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} = n^{-1}\sum_{t=1}^{n} y_t - \hat{\beta}n^{-1}\sum_{t=1}^{n} x_t$. Then, by Lemma 2(a) and part (a) above, as $n \to \infty$,

$$
\begin{aligned}
\frac{\hat{\alpha}}{k_{d_1,n}V_{a,n}} &= \frac{\sum_{t=1}^{n} y_t}{nk_{d_1,n}V_{a,n}} - \left[\frac{k_{d_2,n}V_{b,n}}{k_{d_1,n}V_{a,n}}\hat{\beta}\right]\frac{\sum_{t=1}^{n} x_t}{nk_{d_2,n}V_{b,n}} \\
&\Rightarrow \int_0^1 W_a(r)\mathrm{d}r - \xi_{T2,\beta}\int_0^1 W_b(r)\mathrm{d}r =: \xi_{T2,\alpha}.
\end{aligned}
$$

For part (c), note that

$$
\begin{aligned}
\hat{s}^2 &= \frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{\alpha} - \hat{\beta}x_t)^2 = \frac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2 - \frac{1}{n}\hat{\beta}^2\sum_{t=1}^{n}(x_t - \bar{x})^2 \\
&= \frac{1}{n}\sum_{t=1}^{n} y_t^2 - \left(\frac{\sum_{t=1}^{n} y_t}{n}\right)^2 - \hat{\beta}^2\left[\frac{1}{n}\sum_{t=1}^{n} x_t^2 - \left(\frac{\sum_{t=1}^{n} x_t}{n}\right)^2\right].
\end{aligned}
$$

Then, by Lemma 2(a), Lemma 2(d) and part (a) above, as $n \to \infty$,

$$
\begin{aligned}
\frac{\hat{s}^2}{k_{d_1,n}^2V_{a,n}^2} &= \frac{\sum_{t=1}^{n} y_t^2}{nk_{d_1,n}^2V_{a,n}^2} - \left(\frac{\sum_{t=1}^{n} y_t}{nk_{d_1,n}V_{a,n}}\right)^2 \\
&\quad - \frac{k_{d_2,n}^2V_{b,n}^2}{k_{d_1,n}^2V_{a,n}^2}\hat{\beta}^2\left[\frac{\sum_{t=1}^{n} x_t^2}{nk_{d_2,n}^2V_{b,n}^2} - \left(\frac{\sum_{t=1}^{n} x_t}{nk_{d_2,n}V_{b,n}}\right)^2\right] \\
&\Rightarrow \int_0^1 W_a^2(r)\mathrm{d}r - \left[\int_0^1 W_a(r)\mathrm{d}r\right]^2 \\
&\quad - \xi_{T2,\beta}^2\left\{\int_0^1 W_b^2(r)\mathrm{d}r - \left[\int_0^1 W_b(r)\mathrm{d}r\right]^2\right\} =: \xi_{T2,s^2}.
\end{aligned}
$$

For part (d), note that $t_\beta = \hat{\beta}[\hat{s}^2/\sum_{t=1}^{n}(x_t - \bar{x})^2]^{-1/2} = (\hat{\beta}/\hat{s})[\sum_{t=1}^{n}(x_t - \bar{x})^2]^{1/2}$. Then, by Lemma 2(a), Lemma 2(d) and parts (a) and (c) above, as $n \to \infty$,

$$
\begin{aligned}
\frac{t_\beta}{\sqrt{n}} &= \frac{\hat{\beta}\dfrac{k_{d_2,n}V_{b,n}}{k_{d_1,n}V_{a,n}}}{\dfrac{\hat{s}}{k_{d_1,n}V_{a,n}}}\left[\frac{\sum_{t=1}^{n} x_t^2}{nk_{d_2,n}^2V_{b,n}^2} - \left(\frac{\sum_{t=1}^{n} x_t}{nk_{d_2,n}V_{b,n}}\right)^2\right]^{1/2} \\
&\Rightarrow \frac{\xi_{T2,\beta}}{\xi_{T2,s^2}^{1/2}}\left\{\int_0^1 W_b^2(r)\mathrm{d}r - \left[\int_0^1 W_b(r)\mathrm{d}r\right]^2\right\}^{1/2}.
\end{aligned}
$$

Similarly, by Lemma 2(a), Lemma 2(d) and parts (b)–(c) above, as $n \to \infty$,

$$\begin{aligned}
\frac{t_\alpha}{\sqrt{n}} &= \frac{\hat{\alpha}}{\left\{ n\hat{s}^2 \left[\dfrac{1}{n} + \dfrac{\bar{x}^2}{\sum_{t=1}^n (x_t - \bar{x})^2} \right] \right\}^{1/2}} = \frac{\hat{\alpha}}{\hat{s}} \left[\frac{\sum_{t=1}^n x_t^2 - n^{-1}(\sum_{t=1}^n x_t)^2}{\sum_{t=1}^n x_t^2} \right]^{1/2} \\
&= \frac{\hat{\alpha}/[k_{d_1,n}V_{a,n}]}{\hat{s}/[k_{d_1,n}V_{a,n}]} \left\{ \frac{\dfrac{\sum_{t=1}^n x_t^2}{nk_{d_2,n}^2 V_{b,n}^2} - \left[\dfrac{\sum_{t=1}^n x_t}{nk_{d_2,n}V_{b,n}} \right]^2}{\dfrac{\sum_{t=1}^n x_t^2}{nk_{d_2,n}^2 V_{b,n}^2}} \right\}^{1/2} \\
&\Rightarrow \frac{\xi_{T2,\alpha}}{\xi_{T2,s^2}^{1/2}} \left\{ \frac{\int_0^1 W_b^2(r)\mathrm{d}r - \left[\int_0^1 W_b(r)\mathrm{d}r \right]^2}{\int_0^1 W_b^2(r)\mathrm{d}r} \right\}^{1/2},
\end{aligned}$$

proving (e).

By Lemma 2(a), Lemma 2(d) and part (a) above, as $n \to \infty$,

$$\begin{aligned}
R^2 &= \frac{\sum_{t=1}^n (\hat{y}_t - \bar{y})^2}{\sum_{t=1}^n (y_t - \bar{y})^2} = \hat{\beta}^2 \frac{\sum_{t=1}^n x_t^2 - n^{-1}(\sum_{t=1}^n x_t)^2}{\sum_{t=1}^n y_t^2 - n^{-1}(\sum_{t=1}^n y_t)^2} \\
&= \frac{k_{d_2,n}^2 V_{b,n}^2}{k_{d_1,n}^2 V_{a,n}^2} \hat{\beta}^2 \frac{\dfrac{\sum_{t=1}^n x_t^2}{nk_{d_2,n}^2 V_{b,n}^2} - \left[\dfrac{\sum_{t=1}^n x_t}{nk_{d_2,n}V_{b,n}} \right]^2}{\dfrac{\sum_{t=1}^n y_t^2}{nk_{d_1,n}^2 V_{a,n}^2} - \left[\dfrac{\sum_{t=1}^n y_t}{nk_{d_1,n}V_{a,n}} \right]^2} \\
&\Rightarrow \xi_{T2,\beta}^2 \frac{\int_0^1 W_b^2(r)\mathrm{d}r - \left[\int_0^1 W_b(r)\mathrm{d}r \right]^2}{\int_0^1 W_a^2(r)\mathrm{d}r - \left[\int_0^1 W_a(r)\mathrm{d}r \right]^2},
\end{aligned}$$

proving (f).

Finally, we prove part (g). Recall that $\hat{u}_t = y_t - \hat{\alpha} - \hat{\beta}x_t$ and $\hat{s}^2 = n^{-1}\sum_{t=1}^n \hat{u}_t^2$. Also, note that $k_{d_1,n}$, $k_{d_2,n}$, $V_{a,n}$ and $V_{b,n} \uparrow \infty$ as $n \to \infty$. Then, by Theorem 1(b), Lemma 2(b), Theorem 2(a) and Theorem 2(c), as $n \to \infty$,

$$DW = \frac{\sum_{t=2}^n (\hat{u}_t - \hat{u}_{t-1})^2}{\sum_{t=1}^n \hat{u}_t^2} = \frac{\sum_{t=2}^n (v_t - \hat{\beta}w_t)^2}{\hat{s}^2 n}$$

$$
\begin{aligned}
&= \frac{\frac{\sum_{t=2}^n v_t^2}{n}}{\left[\frac{\hat{s}^2}{k_{d_1,n}^2 V_{a,n}^2}\right] k_{d_1,n}^2 V_{a,n}^2} - \frac{2\frac{k_{d_2,n}V_{b,n}}{k_{d_1,n}V_{a,n}}\hat{\beta}}{\left[\frac{\hat{s}^2}{k_{d_1,n}^2 V_{a,n}^2}\right] k_{d_1,n}V_{a,n}k_{d_2,n}V_{b,n}} \frac{\sum_{t=2}^n v_t w_t}{n} \\
&\quad + \frac{\left[\frac{k_{d_2,n}^2 V_{b,n}^2}{k_{d_1,n}^2 V_{a,n}^2}\hat{\beta}^2\right]}{\left[\frac{\hat{s}^2}{k_{d_1,n}^2 V_{a,n}^2}\right] k_{d_2,n}^2 V_{b,n}^2} \frac{\sum_{t=2}^n w_t^2}{n} \\
&\to_p \quad 0,
\end{aligned}
$$

proving (g). The proof of the theorem is complete. □

Proof of Corollary 1. Note that if $d_1 = d_2 = d$, then $\Theta_{a,0,n} = \Theta_{b,0,n} = \Theta_{0,n}$ and $\Upsilon_{d_1}^2 = \Upsilon_{d_2}^2 = \Upsilon_d^2$ such that $k_{d_1,n} = k_{d_2,n} = k_{d,n}$. Note further that when $\mathbb{E}(a_t^2) = \sigma_a^2 < \infty$ and $\mathbb{E}(b_t^2) = \sigma_b^2 < \infty$, then the law of large numbers gives $n^{-1}\sum_{t=1}^n a_t^2 \to_p \sigma_a^2$ and $n^{-1}\sum_{t=1}^n b_t^2 \to_p \sigma_b^2$, respectively. Given these results, the proof is straightforward and thus omitted. □

Proofs of Corollaries 2–4. Note from Lemma 1 that $\Theta_{a,0,n} \sim n^{d_1}/[\Gamma(d_1)d_1]$, $\Theta_{b,0,n} \sim n^{d_2}/[\Gamma(d_2)d_2]$, $\Upsilon_{d_1}^2 < \infty$ and $\Upsilon_{d_2}^2 < \infty$. It means that $k_{d_1,n} = O(n^{d_1})$ and $k_{d_2,n} = O(n^{d_2})$. Then, the proofs of Corollaries 2–3 are exactly analogous to that of Theorem 2, so we omit them. By the argument just given together with item 5 of Table 3.3 in Banerjee et al. (1993), Corollary 4 holds true obviously. □

5 Concluding remarks

This paper has tried to shed some light on the asymptotic properties of partial sums of fractionally integrated processes which are long memory or, say, long-range dependent. Their applications in studying the asymptotic behaviour of spurious regression problems are given in an explicit way. Our results differ sharply from the ones in the long memory literature where asymptotic distributions are functionals of fractional Brownian motions. From an econometric point of view, the results we obtained are important and interesting in their own right.

Long memory has appeared to be suitable description of the data generating processes for many observed economic and financial variables. As it has been emphasized in Ferson et al. (2003), many of the regressions in the financial literature may be spurious if the dependent variables are persistent and highly autocorrelated regressors are used. On a theoretical level, our results (especially Corollaries 2–4) may offer a better understanding of the simulation results given in the latter paper.

References

Araujo, A. and Gińe, E. (1980). *The Central Limit Theorem for Real and Banach Valued Random Variables*, John Wiley & Sons, New York.

Baillie, R.T. (1996). 'Long memory processes and fractional integration in econometrics', *Journal of Econometrics*, Vol. pp. 73, 5–59.

Banerjee, A., Dolado, J.J., Galbraith, J.W. and Hendry, D.F. (1993). *Co-Integration, Error Correction, and the Econometric Analysis of Non-Stationary Data*, Oxford University Press, Oxford.

Cappuccio, N. and Lubian, D. (1997). 'Spurious regressions between $I(1)$ processes with long memory errors', *Journal of Time Series Analysis*, Vol. 18, pp. 341–354.

Csörgő, M., Szyszkowicz, B. and Wang, Q. (2003). 'Donsker's theorem for self-normalized partial sums processes', *Annals of Probability*, Vol. 31, pp. 1228–1240.

Davydov, Y.A. (1968). 'Convergence of distributions generated by stationary stochastic processes', *Theory of Probability and Its Applications*, Vol. 13, pp. 691–696.

Davydov, Y.A. (1970). 'The invariance principle for stationary processes', *Theory of Probability and Its Applications*, Vol. 15, pp. 487–498.

Doukhan, P. (2003). 'Models, inequalities, and limit theorems for stationary sequences', in Doukhan, P., Oppenheim, G. and Taqqu, M.S. (eds), *Theory and Applications of Long-Range Dependence*. Birkhäuser, Boston, pp. 43–100.

Doukhan, P., Oppenheim, G. and Taqqu, M.S. (2003). *Theory and Applications of Long-Range Dependence*, Birkhäuser, Boston.

Entorf, H. (1997). 'Random walks with drifts: Nonsense regression and spurious fixed-effect estimation', *Journal of Econometrics*, Vol. 80, pp. 287–296.

Feller, W. (1971). *An Introduction to Probability Theory and Its Applications, Vol. II*, 2nd ed., Wiley, New York.

Ferson, W.E., Sarkissian, S. and Simin, T.T. (2003). 'Spurious regressions in Financial Economics?' *Journal of Finance*, Vol. LVIII, pp. 1393–1413.

Gnedenko, B.V. and Kolmogorov, A.N. (1968). *Limit Distributions for Sums of Independent Random Variables*, transl. by K.L. Chung, Addison-Wesley, New York.

Granger, C.W.J., Hyung, N. and Jeon, Y. (2001). 'Spurious regressions with stationary series', *Applied Economics*, Vol. 33, pp. 899–904.

Granger, C.W.J. and Joyeux, R. (1980). 'An introduction to long-memory time series models and fractional differencing', *Journal of Time Series Analysis*, Vol. 1, pp. 15–29.

Granger, C.W.J. and Newbold, P. (1974). 'Spurious regressions in econometrics', *Journal of Econometrics*, Vol. 2, pp. 111–120.

Hall, P. (1992). 'Convergence rates in the central limit theorem for means of autoregressive and moving average sequences', *Stochastic Processes and Their Applications*, Vol. 43, pp. 115–131.

Hall, P. and Heyde, C.C. (1980). *Martingale Limit Theory and Its Application*, Academic Press, New York.

Henry, M. and Zaffaroni, P. (2003). 'The long-range dependence paradigm for macroeconomics and finance', in Doukhan, P., Oppenheim, G., and Taqqu, M.S. (eds), *Theory and Applications of Long-Range Dependence*, Birkhäuser, Boston, pp. 417–438.

Hosking, J.R.M. (1981). 'Fractional differencing', *Biometrika*, Vol. 68, pp. 165–176.

Kim, T.-H., Lee, Y.-S. and Newbold, P. (2004). 'Spurious regressions with stationary processes around linear trends', *Economics Letters*, Vol. 83, pp. 257–262.

Marmol, F. (1995). 'Spurious regressions between $I(d)$ processes', *Journal of Time Series Analysis*, Vol. 16, pp. 313–321.

Marmol, F. (1996). 'Nonsense regressions between integrated processes of different orders', *Oxford Bulletin of Economics and Statistics*, Vol. 58, pp. 525–536.

Marmol, F. (1998). 'Spurious regression theory with nonstationary fractionally integrated processes', *Journal of Econometrics*, Vol. 84, pp. 233–250.

Marmol, F. and Reboredo, J.C. (1999). 'Near observational equivalence and fractionally integrated processes', *Oxford Bulletin of Economics and Statistics*, Vol. 61, pp. 283–290.

Molinas, C. (1986). 'A note on spurious regressions with integrated moving average errors', *Oxford Bulletin of Economics and Statistics*, Vol. 48, pp. 279–282.

Peligrad, M. (1982). 'Invariance principles for mixing sequences of random variables', *Annals of Probability*, Vol. 10, pp. 968–981.

Petrov, V.V. (1995). *Limit Theorems of Probability Theory*, Clarendon Press, Oxford.

Phillips, P.C.B. (1986). 'Understanding spurious regressions in econometrics', *Journal of Econometrics*, Vol. 33, pp. 311–340.

Phillips, P.C.B. (1987). 'Time series regression with a unit root', *Econometrica*, Vol. 55, 277–301.

Robinson, P.M. (1994). 'Time series with strong dependence', in Sims, C.A. (ed), *Advances in Econometrics, Vol. 1*, Cambridge University Press, Cambridge, pp. 47–95.

Robinson, P.M. (2003). *Time Series with Long Memory*, Oxford University Press, Oxford.

Stout, W.F. (1974). *Almost Sure Convergence*, Academic, New York.

Taqqu, M.S. (2003). 'Fractional Brownian motion and long-range dependence', in Doukhan, P., Oppenheim, G. and Taqqu, M.S. (eds), *Theory and Applications of Long-range Dependence*, Birkhäuser, Boston, pp. 5–38.

Tsay, W.-J. and Chung, C.-F. (2000). 'The spurious regression of fractionally integrated processes', *Journal of Econometrics*, Vol. 96, pp. 155–182.