Dynamic Moral Hazard with History-Dependent Reservation Utilities*

Stanimir Morfov†

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Abstract

This paper considers a moral hazard problem in an infinite-horizon, principal-agent framework characterized by limited commitment and history-dependent reservation utilities. I prove existence and construct a reduced equivalent representation of the problem that can be addressed by numerical techniques. In computing the endogenous state space, I use an innovative algorithm which does not rely on the convexity of the underlying set. Further on, I focus on the estimation of the dynamically optimal compensation for US executives and find evidence that in the presence of positive correlation between stock prices and reservation utilities, the contract provides the CEO with insurance against bad outcomes, which ultimately smooths his/her consumption across (initial-history) states. Exerting effort appears to be the predominant strategy for the principal, but shirking may still be optimal when the agent is rich enough. The optimal wage scheme and the future utility of the CEO tend to grow in both his/her current utility and in the future realization of the stock price. The agent's utility weakly increases in the long run.

Keywords: principal-agent problem, moral hazard, dynamic contracts, executive compensation

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†State University - Higher School of Economics, International College of Economics and Finance, Pokrovski Bulvar, 11, 109028 Moscow, Russia, smorfov@hse.ru

1 Introduction

During the last years, there has been a revived interest in the theory of dynamic contracting[1]. However, although most of the research incorporates some form of limited commitment/enforcement, little has been done in terms of extending the notion of commitment per se. In particular, there is no reason to believe that the outside option is constant across the history of observables. For example, it is unrealistic to treat the reservation utility of a CEO as fixed regardless of the situation in his/her firm, industry, or the economy as a whole. The dependence could come through many channels- externalities, different types of agents, a certain structure of beliefs, but more importantly, it can significantly influence the nature of the relationship and the form of the optimal contract. Moreover, extending the notion of commitment can bring some important insights into various contractual problems. For example, in order to address the wide use of broad-based stock option plans, Oyer (2004) builds a simple 2-period model where adjusting compensation is costly and employee's outside opportunities are correlated with the firm's performance.

In this sense, what remains to be done is to generalize the notion of commitment by defining the outside options on the history observed in a dynamic contractual setting.

The current paper considers a moral hazard problem in an infinitely repeated principal-agent interaction while allowing the reservation utilities of both parties to vary across the history of observables. More precisely, to keep the model tractable, the reservation utilities are assumed to depend on some finite truncation of the publicly observed history. The rest of the model is standard in the sense that the principal wants to implement some sequence of actions which stochastically affect a variable of his/her interest, but suffers from the fact that the actions are unobservable. For this purpose, the optimal contract needs to provide the proper incentives for the agent to exercise the sequence of actions suggested by the principal. The incentives, however, are restricted by the inability of the parties to commit to a long-term relationship. It is here where the dynamics of the reservation utilities enters the relationship by reshaping the set of possible self-enforcing, incentive-compatible contracts.

In order to be able to characterize the optimal contract in such a setting, I construct a reduced stationary representation of the model in line with the dynamic insurance literature. The representation benefits from Green (1987)- the notion of temporary incentive compatibility, Spear and Srivastava (1987)- the recursive formulation of the problem with the agent's expected discounted utility taken as the state variable, and Phelan (1995)- the recursive structure with limited commitment, but is closest to Wang (1997) as far as the recursive form is concerned. Unlike Wang (1997), however, I formally introduce limited commitment on both sides and provide a rigorous treatment of its effect on the

[1] See, for example, Fernandes and Phelan (2000), Ligon, Thomas and Worrall (2000), Wang (2000), Phelan and Stacchetti (2001), Sleet and Yeltekin (2001), Ligon, Thomas and Worrall (2002), Ray (2002), Thomas and Worrall (2002), Doepke and Townsend (2004), Jarque (2005).

structure of the reduced computable version of the model. A parallel research by Aseff (2004) uses a similar general formulation[2], but via a transformation due to Grossman and Hart (1983) constructs a dual, cost-minimizing recursive form closer to Phelan (1995) in order to solve for the optimal contract. Such a procedure, however, exogenously imposes the optimality of a certain action on every possible contingency.

After existence is proved, the general form of the model is reduced to a more tractable, recursive form where the state is given by the agent's (promised) expected discounted utility. On a different dimension, the state space includes the set of truncated initial price histories in order to account for their influence[3] on the reservation utilities. This recursive formulation does not rely on the first-order approach and is not based on Lagrange multipliers [cf. Marcet and Marimon, (1998)]. In fact, all I need is continuity of the momentary utilities. I first consider an auxiliary version where the participation of the principal is not guaranteed. The solution of this problem can be computed through standard dynamic programming methods once the state space is determined. Following the approach of Abreu, Pearce and Stacchetti (1990), the state space is shown to be the fixed point of a set operator and can be obtained through successive iteration on this operator until convergence. Given the solution of the auxiliary problem, I resort to a procedure outlined by Rustichini (1998) in order to solve for the optimal incentive compatible, two-side participation guaranteed supercontract. This is achieved by severely punishing the principal for any violation of his/her participation constraint. The procedure allows of recovering the subspace of agent's expected discounted utilities supportable by a self-enforcing incentive-compatible contract.

Regarding the numerical computation, one point deserves special attention. In computing the endogenous state space we are iterating on sets and therefore need to represent them efficiently. For the class of infinitely repeated games with perfect monitoring, Judd, Yeltekin and Conklin (2003) are able to construct inner and outer convex polytope approximations based on the convexification of the equilibrium value set through a public randomization device. The algorithm I use may be of independent interest since it does not rely on the convexity of the underlying set. The main idea is to discretize the guess for the equilibrium set elementwise, extract small open balls around the gridpoints unfeasible with respect to the (non-updated) guess and use the remaining set, i.e. the guess less the extracted intervals, as a new guess for the equilibrium set. The procedure stops if the structure of the representations of two successive guesses coincides[4] and the suitably defined difference between the representations is less than some prespecified tolerance level.

The results suggest that with a loose upper bound on wages, the optimal contract can support extremely high values for the expected discounted utility of the

[2] His benchmark model is a full-commitment one, but he considers limited commitment on part of the agent as an extension.

[3] The relationship between price histories and reservation utilities is predetermined since the reservation utilities are exogenous to the problem.

[4] Namely, if the representations have the same number of closed sets element by element.

CEO when the participation of the principal is not guaranteed. However, when solving for the self-enforcing contract, these values naturally disappear since they violate principal's participation constraint. In case of positive correlation between firm's stock prices and agent's reservation utilities, the minimum utility the CEO can be promised for initial histories characterized by lower reservation utility is boosted by higher reservation utilities for other states. This suggests that the participation constraint of the agent does not bind in states characterized by low stock prices. In other words, the optimal contract provides the agent with some insurance against bad outcomes, which ultimately smooths his/her consumption across (initial history) states. Exerting effort appears to be the predominant strategy for the principal, but shirking may still be optimal when the agent is rich enough. The optimal wage scheme and the future utility of the agent tend to grow in both current utility and in the future realization of the stock price. The CEO's utility weakly increases in the long run. In particular, agents who start rich tend to keep their utility level while those who start poor get richer in time.

The rest of the paper is structured as follows. Section 2 presents the general dynamic model. Section 3 derives the reduced recursive formulation. Section 4 explains the numerical algorithm at a practical level and discusses the results. Section 5 concludes. Appendix 1 contains all the proofs. Appendix 2 presents the results.

2 Dynamic model

The model considers a moral hazard problem in an infinite horizon principal-agent framework with limited commitment on both sides. Before specifying it formally, I introduce some notation.

Let $\mathbb{Z}$ denote the set of integers and define $\overline{\mathbb{Z}} := \mathbb{Z} \cup \{\pm\infty\}$ as its two-point compactification. Let $\overline{\mathbb{Z}}_{++} := \left\{z \in \overline{\mathbb{Z}} : z > 0\right\}$ and $\overline{\mathbb{Z}}_{+} := \overline{\mathbb{Z}}_{++} \cup \{0\}$ denote the sets of positive and respectively nonnegative integers. $\overline{\mathbb{Z}}_{--}$ and $\overline{\mathbb{Z}}_{-}$ are defined accordingly. Let y_t denote the firm's stock price in the end of period t, $\forall t \in \overline{\mathbb{Z}}$. Then for $\forall t, \tau \in \overline{\mathbb{Z}} : t \leq \tau$, define ${}^{t}y^{\tau} := (y_t, y_{t+1}, ..., y_\tau)$ as the stream of prices from period t to period τ. For $\forall t \in \overline{\mathbb{Z}}_{--}$, let ${}^{t}y := {}^{t}y^{-1}$ and for $\forall \tau \in \overline{\mathbb{Z}}_{+}$, let $y^{\tau} := {}^{0}y^{\tau}$. The set of possible stock prices is assumed a stationary, finite subset of the nonnegative real line, namely $\forall t \in \overline{\mathbb{Z}}$, $y_t \in Y := \left\{y_{(1),...,}y_{(N)}\right\} \subset \mathbb{R}_{+}$, where $\underline{y} = y_{(1)} \leq y_{(2)} \leq ... \leq y_{(N-1)} \leq y_{(N)} = \overline{y}$ with $N \in \mathbb{Z}_{++} \setminus \{1\}$ and $\underline{y} < \overline{y}$. Henceforth, I adopt the convention that for $\forall T \in \overline{\mathbb{Z}}_{+}$ and an arbitrary set W, $W^{T} := \mathop{\times}\limits_{i=1}^{T} W$ with $W^{0} := \emptyset$. For example, Y^{T} denotes the set of possible price streams of length T periods.

The timing is as follows. At the beginning of each period $\tau \in \overline{\mathbb{Z}}_{+}$, after a particular stock-price history ${}^{-\infty}y^{\tau-1}$ has been publicly observed, a contract $c_\tau\left({}^{-\infty}y^{\tau-1}\right) := \left(a_\tau\left({}^{-\infty}y^{\tau-1}\right), w_\tau\left({}^{-\infty}y^{\tau-1}, Y\right)\right)$ is signed between a risk neutral principal (staying for the firm's shareholders)[5] and a risk-averse agent (CEO). The contract specifies an action $a_\tau\left({}^{-\infty}y^{\tau-1}\right)$ to be implemented by the agent. The action here stays for the level of effort the agent should exert on the job. To make the analysis tractable, the action is assumed one-dimensional and the action space is taken compact, time and history invariant. Formally, $a_\tau\left({}^{-\infty}y^{\tau-1}\right) \in A$, where $A \subset \mathbb{R}$ compact. Let $\underline{a} = \min\{A\}$ and $\overline{a} = \max\{A\}$ and assume $\underline{a} < \overline{a}$. The contract also specifies a compensation scheme $w_\tau\left({}^{-\infty}y^{\tau-1}, Y\right)$, under which the agent will receive a monetary payoff $w_\tau\left({}^{-\infty}y^{\tau-1}, y_\tau\right) \geq \underline{w} \subset \mathbb{R}$ in the end of the period if the firm's (end-of-period) stock price is y_τ, $\forall y_\tau \in Y$. After the contract is signed, the agent exercises action $a_\tau\left({}^{-\infty}y^{\tau-1}\right)$. Then, y_τ is observed and the agent receives $w_\tau\left({}^{-\infty}y^{\tau-1}, y_\tau\right)$. At the beginning of period $\tau+1$, contract $c_{\tau+1}\left({}^{-\infty}y^{\tau}\right)$ is signed and so on.

One can interpret the exogenous lower bound for the wage, $\underline{w}$, as a minimum wage, but in order to keep the analysis as general as possible I abstract from imposing any specific restrictions on it.

While the agent's action is unobservable by the principal, it influences the realization of the stock price, but in a non-deterministic way. For $\forall \xi \in \mathbb{R}$, $\forall a_t\left({}^{-\infty}y^{t-1}\right) \in A^\infty \times A^t$, $\forall {}^{-\infty}y^{t-1} \in Y^\infty \times Y^t$, $\forall t \in \overline{\mathbb{Z}}_+$, let $f\left(\xi | a^t\left({}^{-\infty}y^{t-1}\right), {}^{-\infty}y^{t-1}\right)$ be the probability that y_t equals ξ after a history ${}^{-\infty}y^{t-1}$ has been observed and an action sequence $a^t\left({}^{-\infty}y^{t-1}\right)$ has been implemented. Then, starting from any node we can construct the probability of any future contingency by recursively applying Bayes' rule.

In order to be able to introduce an efficiently computable form of the model, I assume that f depends only on the action taken in the current period and each stock price in the admissible set Y is reached with a strictly positive probability.

Assumption 1: For $\forall a_t\left({}^{-\infty}y^{t-1}\right) \in A$, $\forall {}^{-\infty}y^{t-1} \in Y^{t-1}$, $\forall t \in \overline{\mathbb{Z}}_+$, $f\left(.|a^t\left({}^{-\infty}y^{t-1}\right), {}^{-\infty}y^{t-1}\right) = \pi\left(.|a_t\left({}^{-\infty}y^{t-1}\right)\right)$, with $\pi\left(y|a_t\left({}^{-\infty}y^{t-1}\right)\right) > 0$, $\forall y \in Y$, $\pi\left(y|a_t\left({}^{-\infty}y^{t-1}\right)\right) = 0$ otherwise, and $\sum_{y \in Y} \pi\left(y|a_t\left({}^{-\infty}y^{t-1}\right)\right) = 1$. Moreover, for $\forall y \in Y$, $\pi(y|.)$ is continuous on A.

The second part of the assumption is a regularity condition which is trivially satisfied if A is finite.

Since the principal is risk-neutral, without loss of generality his/her (end-of-) period- utility can be assumed equal to the firm's stock price less the agent's compensation. He/she discounts the future by a factor $\beta_P \in (0,1)$. The agent's period utility function is assumed separable in monetary payoff and effort. More specifically, after a history $y^\tau = \left({}^{-\infty}y^{\tau-1}, y_\tau\right)$ has been observed, the agent's (end-of-) period utility is given by $v\left(w_\tau\left({}^{-\infty}y^{\tau-1}, y_\tau\right)\right) - a_\tau\left({}^{-\infty}y^{\tau-1}\right)$, where $v : \mathbb{R} \to \mathbb{R}$ is assumed twice continuously differentiable with $v'(.) > 0$ and $v''(.) < 0$. The agent discounts the future by a factor $\beta_A \in (0,1)$.

In the analysis so far, it was implicitly assumed that the agent exerts the level of effort specified in the contract. However, this is not necessarily so, since

[5] No conflicts of interest are assumed between the firm's shareholders and the principal.

if another action brings the agent strictly higher utility, he/she will find it profitable to deviate. Therefore, the contract should provide the proper incentives to the agent in order for him/her to exercise exactly the action recommended by the principal.

Limited commitment is assumed on both parts in the sense that both the principal and the agent can commit only to short-term (single-period) contracts. This assumption is intended to reflect legal issues on the enforcement of long-term contracts. However, at the initial period the principal can offer a long term contract (a supercontract) that neither he/she, nor the agent would like to renege on, and that would provide the necessary incentives for the agent to exercise the sequence of actions proposed by the principal.

In order to introduce the issue of commitment, we should specify reservation utilities for the agent and the principal. For $\forall {}^{-\infty}y^{\tau-1} \in Y^{\infty} \times Y^{\tau}$, $\tau \in \overline{\mathbb{Z}}_{+}$, let $\underline{V}\left({}^{-\infty}y^{\tau-1}\right)$, $\underline{U}\left({}^{-\infty}y^{\tau-1}\right) \in \mathbb{R}$ be the reservation utilities (in expected discounted terms) of the agent and respectively the principal after a history ${}^{-\infty}y^{\tau-1}$. Since it is not practical to define reservation utilities on infinite histories, I make the following assumption.

Assumption 2: $\exists \theta \in \mathbb{Z}_{+} : \forall \tau \in \overline{\mathbb{Z}}_{+}, \forall {}^{-\infty}y'^{\tau-1} \in Y^{\infty} \times Y^{\tau} : {}^{\tau-\theta}y'^{\tau-1} = {}^{-\theta}y$, $\underline{V}\left({}^{-\infty}y'^{\tau-1}\right) = \underline{V}_{{}^{-\theta}y} \in \left[\underline{V}, \overline{V}\right] \subset \mathbb{R}$ and $\underline{U}\left({}^{-\infty}y'^{\tau-1}\right) = \underline{U}_{{}^{-\theta}y} \in \left[\underline{U}, \overline{U}\right] \subset \mathbb{R}$, with $\underline{V}\left({}^{-\infty}y'^{\tau-1}\right) = \underline{V}$ and $\underline{U}\left({}^{-\infty}y'^{\tau-1}\right) = \underline{U}$ if $\theta = 0$.

The assumption says that the reservation utilities are finite-history dependent, but time independent. Note that the history dependence is truncated to the prices in last θ periods for both the principal and the agent. This is done for the purpose of simplifying the notation. If we have $\theta_V \neq \theta_U$, we can take $\theta := \theta_V \vee \theta_U$ and the analysis will not change.[6]

Given Assumptions 1 and 2, nothing in the problem being solved by the principal in period 0 depends on ${}^{-\infty}y^{-\theta-1}$, the initial history before the previous θ stock price realizations. Therefore, it would be convenient to restrict the possible initial histories to the ones contained in the set Y^{θ}. Henceforth, I will refer to a general element of this set as a truncated history. For $\forall y^{-\theta} \in Y^{\theta}$ let $c_{{}^{-\theta}y} := \left(a_{{}^{-\theta}y}, w_{{}^{-\theta}y}\right)$, where $a_{{}^{-\theta}y} := \left\{\left\{a_t\left({}^{-\theta}y^{t-1}\right) : y^{t-1} \in Y^t\right\}\right\}_{t=0}^{\infty}$ and $w_{{}^{-\theta}y} := \{\{w_t\left(\left({}^{-\theta}y^{t-1}, y_t\right)\right) : \left(y^{t-1}, y_t\right) \in Y^{t-1} \times Y\}\}_{t=0}^{\infty}$ are respectively the plan of actions and the sequence of wages defined on any contingency given an initial truncated history ${}^{-\theta}y \in Y^{\theta}$. Now, for $\forall {}^{-\theta}y^{\tau-1} \in {}^{-\theta}y \times Y^{\tau}$, $\forall \tau \in \mathbb{Z}_{+}$, $\forall {}^{-\theta}y \in Y^{\theta}$, define $V_{\tau}\left(c_{{}^{-\theta}y}, {}^{-\theta}y^{\tau-1}\right) :=$

$$= \sum_{t=\tau}^{\infty} \beta_A^{t-\tau} \sum_{y_t \in Y} \ldots \sum_{y_\tau \in Y} \left[\upsilon\left(w_t\left({}^{-\theta}y^t\right)\right) - a_t\left({}^{-\theta}y^{t-1}\right)\right] \prod_{i=\tau}^{t} \pi\left(y_i | a_i\left({}^{-\theta}y^{i-1}\right)\right) \text{ and}$$

$$U_{\tau}\left(c_{{}^{-\theta}y}, {}^{-\theta}y^{\tau-1}\right) := \sum_{t=\tau}^{\infty} \beta_P^{t-\tau} \sum_{y_t \in Y} \ldots \sum_{y_\tau \in Y} \left[y_t - w_t\left({}^{-\theta}y^t\right)\right] \prod_{i=\tau}^{t} \pi\left(y_i | a_i\left({}^{-\theta}y^{i-1}\right)\right)$$

[6] The only gain of introducing θ_V and θ_U is in the case of $\theta_V < \theta_U$ because then we can define the operator T on Y^{θ_V} instead of on $Y^{\theta_V \vee \theta_U}$, thus improving the efficiency of the numerical estimation. However, since all the results are analogous, I prefer working with θ.

as the expected discounted utilities of the agent and respectively the principal at node ${}^{-\theta}y^{\tau-1}$.

At time 0, after a truncated history ${}^{-\theta}y \in Y^{\theta}$ has been observed, the principal is solving the following problem:

[PP]

$$U^{**}_{-\theta y} := \sup_{c_{-\theta y}} U_0\left(c_{-\theta y}, {}^{-\theta}y\right) \text{ s.t.:}$$

$$a_t\left({}^{-\theta}y^{t-1}\right) \in A,\ \forall {}^{-\theta}y^{t-1} \in {}^{-\theta}y \times Y^t,\ \forall t \in \mathbb{Z}_+ \tag{1}$$

$$w_t\left({}^{-\theta}y^{t}\right) \geq \underline{w},\ \forall {}^{-\theta}y^{t} \in {}^{-\theta}y \times Y^{t+1},\ \forall t \in \mathbb{Z}_+ \tag{2}$$

$$V_0\left(a_{-\theta y}, w_{-\theta y}, {}^{-\theta}y\right) \geq V_0\left(a'_{-\theta y}, w_{-\theta y}, {}^{-\theta}y\right),$$
$$\forall a'_{-\theta y} : a'_t\left({}^{-\theta}y^{t-1}\right) \in A,\ \forall {}^{-\theta}y^{t-1} \in {}^{-\theta}y \times Y^t,\ \forall t \in \mathbb{Z}_+ \tag{3}$$

$$V_\tau\left(c_{-\theta y}, {}^{-\theta}y^{\tau-1}\right) \geq \underline{V}_{-\theta y},\ \forall {}^{-\theta}y^{\tau-1} \in {}^{-\theta}y \times Y^{\tau},\ \forall \tau \in \mathbb{Z}_+ \tag{4}$$

$$U_\tau\left(c_{-\theta y}, {}^{-\theta}y^{\tau-1}\right) \geq \underline{U}_{-\theta y},\ \forall {}^{-\theta}y^{\tau-1} \in {}^{-\theta}y \times Y^{\tau},\ \forall \tau \in \mathbb{Z}_+ \tag{5}$$

Constraint (1) guarantees that the action plan is admissible. (2) keeps the wage schedule above its exogenously given lower bound. (3) is a period-0 incentive compatibility constraint requiring that the action plan of the principal should make the agent weakly better off in terms of period-0 expected discounted utility than any other sequence of admissible actions. Constraint (4) is the participation constraint, which due to the limited commitment on part of the agent should hold every period. Constraint (5) is required by the fact that the principal cannot commit to a long-term contract and guarantees his/her participation.

Let $\Gamma_{-\theta y} := \left\{c_{-\theta y} : (1) - (5) \text{ hold}\right\}$ denote the set of constraints given an initial truncated history ${}^{-\theta}y$.

Assumption 3: $\forall {}^{-\theta}y \in Y^{\theta}$, $\Gamma_{-\theta y} \neq \emptyset$.

3 Recursive Form

Proposition 1: *Let (1), (2) hold after* $^{-\theta}y \in Y^{\theta}$ *and* $w_t\left(^{-\theta}y^t\right) \leq \overline{w}$, $\forall^{-\theta}y^t \in {}^{-\theta}y \times Y^{t+1}$, $\forall t \in \mathbb{Z}_+$ *for some* $\overline{w} \in \mathbb{R}$. *Then, (3)* $\Leftrightarrow$

$$
\begin{aligned}
&\forall^{-\theta}y^{\tau-1} \in^{-\theta} y \times Y^{\tau}, \forall\tau \in \mathbb{Z}_+, \\
V_\tau\left(a_{-\theta y}, w_{-\theta y}, ^{-\theta} y^{\tau-1}\right) &\geq V_\tau\left(a'_{-\theta y}, w_{-\theta y}, ^{-\theta} y^{\tau-1}\right), \forall a'_{-\theta y} : a'_t\left(^{-\theta}y^{t-1}\right) \in A, \\
&\forall^{-\theta}y^{t-1} \in \left(^{-\theta}y, y^{\tau-1}\right) \times Y^{t-\tau}, \forall t \in \mathbb{Z}_+ : t \geq \tau
\end{aligned} \tag{6}
$$

This proposition shows that incentive compatibility at an initial node $^{-\theta}y$ is equivalent to incentive compatibility at all the nodes following $^{-\theta}y$. The uniform upper bound on the wage is introduced in order to guarantee the boundedness of the discounted expected utility of the agent at every node.

Proposition 2: *Let (1), (2) and (5) hold after* $^{-\theta}y \in Y^{\theta}$ *and* $w_t\left(^{-\theta}y^t\right) \leq \overline{w}$, $\forall^{-\theta}y^t \in {}^{-\theta}y \times Y^{t+1}$, $\forall t \in \mathbb{Z}_+$. *Then, (3)* $\Leftrightarrow$

$$
\begin{aligned}
&\forall^{-\theta}y^{\tau-1} \in {}^{-\theta}y \times Y^{\tau}, \forall\tau \in \mathbb{Z}_+, \\
&V_\tau\left(a_{-\theta y}, w_{-\theta y}, ^{-\theta} y^{\tau-1}\right) \geq V_\tau\left(a'_{-\theta y}, w_{-\theta y}, ^{-\theta} y^{\tau-1}\right), \\
&\forall a'_{-\theta y} : a'_\tau\left(^{-\infty}y^{\tau-1}\right) \in A,\ a'_t\left(^{-\infty}y^{t-1}\right) = a_t\left(^{-\infty}y^{t-1}\right), \forall^{-\infty}y^{t-1} \in \\
&\left(^{-\infty}y, y^{\tau-1}\right) \times Y^{t-\tau}, \forall t \in \mathbb{Z}_{++} : t > \tau
\end{aligned} \tag{7}
$$

The proposition says that constraint (3) is equivalent to requiring that after any history $^{-\infty}y^{\tau-1} \in {}^{-\infty}y \times Y^{\tau}$, at any date $\tau \in \mathbb{Z}_+$ there is no profitable deviation in the current period which will make the agent strictly better off (in expected utility terms) given that he/she fully complies to the plan in the future (Green (1987)'s temporary incentive compatibility). The uniform ceiling on wages serves the same purpose as in Proposition 1.

Proposition 3: *Let (1), (2) and (5) hold after some* $^{-\theta}y \in Y^{\theta}$. *Then for* $\forall^{-\theta}y^t \in {}^{-\theta}y \times Y^{t+1}$, $\forall t \in \mathbb{Z}_+$, $w\left(^{-\theta}y^t\right) \leq \overline{w}$, *where* $\overline{w} := \frac{1}{\underline{\pi}}\left(\frac{\overline{y}-\underline{w}}{1-\beta_P} - \underline{U}\right) + \underline{w}$ *with* $\underline{\pi} := \min\limits_{(y,a)\in Y\times A} \pi\left(y|a\right)$.

This proposition says that an admissible contract that guarantees the commitment of the principal effectively binds the wage from above. Consequently, the results of Propositions 1 and 2 apply to such a contract. In particular, they are true for any contract in the constrained set $\Gamma_{-\theta_y}$ of the problem [PP]. Finally, note that $\overline{w}$ does not depend on the initial truncated history ${}^{-\theta}y$.

Take an arbitrary $\tau \in \mathbb{Z}_+$ and history ${}^{-\theta}y^{\tau-1} \in Y^{\theta+\tau}$ and let $\left\{V_\tau^{IC2P}\left({}^{-\theta}y^{\tau-1}\right)\right\} := \{V \in \mathbb{R} : \exists\,(a,w)\,|(1),(2),(4),(5),(7)$ hold after a history ${}^{-\theta}y^{\tau-1}$ and $V_\tau\left(a,w,{}^{-\theta}y^{\tau-1}\right) = V\}$ be the set of admissible values for the expected discounted utility of the agent signing at date τ after a history ${}^{-\theta}y^{\tau-1}$ an incentive-compatible supercontract that guarantees both his/her participation and that of the principal (an IC2P contract). For the purposes of estimation, we introduce another set $\left\{V_\tau^{ICAP}\left({}^{-\theta}y^{\tau-1}\right)\right\} := \{V \in \mathbb{R} : \exists\,(a,w)\,|(1),(2'),(4),(7)$ hold after a history ${}^{-\theta}y^{\tau-1}$ and $V_\tau\left(a,w,{}^{-\theta}y^{\tau-1}\right) = V\}$, where $(2')$ stays for $w_t\left({}^{-\theta}y^t\right) \in [\underline{w},\overline{w}]$, $\forall {}^{-\theta}y^t \in {}^{-\theta}y \times Y^{t+1}$, $\forall t \in \mathbb{Z}_+$ with $\overline{w}$ defined as in Proposition 3. This set gives us the possible discounted utilities of the agent signing at date τ after a history ${}^{-\theta}y^{\tau-1}$ an incentive-compatible contract that guarantees the participation of the agent, but not that of the principal and places a ceiling $\overline{w}$ on the agent's salary at every node (an ICAP contract). Note that by Proposition 3 and construction, $\forall {}^{-\theta}y^{\tau-1} \in Y^{\theta+\tau}$, $\forall \tau \in \mathbb{Z}_+$, $\left\{V_\tau^{IC2P}\left({}^{-\theta}y^{\tau-1}\right)\right\} \subset \left\{V_\tau^{ICAP}\left({}^{-\theta}y^{\tau-1}\right)\right\}$.

For $\forall V \in \left\{V_\tau^{IC2P}\left({}^{-\theta}y^{\tau-1}\right)\right\}$, let $\left\{U_\tau^{IC2P}\left(V,{}^{-\theta}y^{\tau-1}\right)\right\} := \{U \in \mathbb{R} : \exists\,(a,w)\,|$ $(1),(2),(4),(5),(7)$ hold after ${}^{-\theta}y^{\tau-1}$, $V_\tau\left(a,w,{}^{-\theta}y^{\tau-1}\right) = V$ and $U_\tau\left(a,w,{}^{-\theta}y^{\tau-1}\right) = U\}$ be the set of possible values for the expected discounted utility of the principal signing at node ${}^{-\theta}y^{\tau-1}$ at time τ an IC2P supercontract that would give the agent an initial expected discounted utility of V. For $\forall V \in \left\{V_\tau^{ICAP}\left({}^{-\theta}y^{\tau-1}\right)\right\}$, let $\left\{U_\tau^{ICAP}\left(V,{}^{-\theta}y^{\tau-1}\right)\right\}$ be the corresponding set (defined accordingly) in case the principal is signing an ICAP contract instead. For $\forall V \in \left\{V_\tau^{IC2P}\left({}^{-\theta}y^{\tau-1}\right)\right\}$, we have $\left\{U_\tau^{IC2P}\left(V,{}^{-\theta}y^{\tau-1}\right)\right\} \subset \left\{U_\tau^{ICAP}\left(V,{}^{-\theta}y^{\tau-1}\right)\right\}$, while for $V \in \left\{V_\tau^{ICAP}\left({}^{-\theta}y^{\tau-1}\right)\right\} \setminus \left\{V_\tau^{IC2P}\left({}^{-\theta}y^{\tau-1}\right)\right\}$, $\left\{U_\tau^{IC2P}\left(V,{}^{-\theta}y^{\tau-1}\right)\right\}$ is not defined.

Proposition 4: *For $i \in \{IC2P, ICAP\}$, we have: (a) $\left\{V_\tau^i\left({}^{-\theta}\widetilde{y}^{\tau-1}\right)\right\} = \left\{V^i\left({}^{-\theta}y\right)\right\}$, $\forall {}^{-\theta}\widetilde{y}^{\tau-1} \in Y^\tau \times {}^{-\theta}y$, $\forall \tau \in \mathbb{Z}_+$ with $\left\{V^i\left({}^{-\theta}y\right)\right\}$ compact, $\forall {}^{-\theta}y \in Y^\theta$; (b) $\forall V^i \in \left\{V_\tau^i\left({}^{-\theta}\widetilde{y}^{\tau-1}\right)\right\}$, $\forall {}^{-\theta}\widetilde{y}^{\tau-1} \in Y^\tau \times {}^{-\theta}y$, $\forall \tau \in \mathbb{Z}_+$, $\left\{U_\tau^i\left(V^i,{}^{-\theta}\widetilde{y}^{\tau-1}\right)\right\} = \left\{U^i\left(V^i,{}^{-\theta}y\right)\right\}$ compact, $\forall {}^{-\theta}y \in Y^\theta$.*

Part (a) of the proposition says that the sets of possible expected discounted utility values for the agent signing an IC2P or ICAP contract are time invariant and compact. Furthermore, the history dependence of these sets is restricted only to the previous (as of signing) θ stock price realizations. As part (b) indicates, the results also apply to the set of possible expected discounted utilities of the principal signing an IC2P or ICAP contract guaranteeing a particular initial utility to the agent. For the sake of consistency, unless explicitly specified, any of these sets will be regarded as after a truncated history ${}^{-\theta}y \in Y^\theta$ in period 0.

Proposition 5 *(Existence of an optimal contract): For $\forall {}^{-\theta}y \in Y^\theta$, $\exists\left(a^{**}_{-\theta_y}, w^{**}_{-\theta_y}\right) \in \Gamma_{-\theta_y}$ s.t. $U^{**}_{-\theta_y} = U_0\left(a^{**}_{-\theta_y}, w^{**}_{-\theta_y}, {}^{-\theta}y\right)$.*

The proposition establishes the existence of an optimal IC2P supercontract. However, due to the complexity of the problem, the optimal contract cannot be computed analytically. Therefore, I resort to numerical methods. First, the original problem [PP] has to be given a computable representation. In the spirit of Spear and Srivastava (1987), this is done by constructing a recursive version of [PP] taking the agent's expected discounted utility as a state variable. Up to certain qualifications, this new formulation of the problem can be addressed by dynamic programming routines.

Before introducing the recursive form, I establish some useful results regarding the transition to the new state variable. Namely, I show that it does not affect the optimal solution.

Fix ${}^{-\theta}y \in Y^{\theta}$. By Proposition 4 (b) for $\forall V \in \left\{V^{IC2P}\left({}^{-\theta}y\right)\right\}$, $\left\{U^{IC2P}\left(V,{}^{-\theta}y\right)\right\}$ is compact and therefore, we can define $U^{*}\left(V,{}^{-\theta}y\right) := \max\left\{U^{IC2P}\left(V,{}^{-\theta}y\right)\right\}$ as the maximum utility the principal can get by signing an IC2P supercontract offering V to the agent. Furthermore, let $U^{*}_{{}^{-\theta}y} := \sup\limits_{V\in\left\{V^{IC2P}\left({}^{-\theta}y\right)\right\}} U^{*}\left(V,{}^{-\theta}y\right)$.

Proposition 6: *For* $\forall {}^{-\theta}y \in Y^{\theta}$, $U^{**}_{{}^{-\theta}y} = U^{*}_{{}^{-\theta}y} = \max\limits_{V\in\left\{V^{IC2P}\left({}^{-\theta}y\right)\right\}} U^{*}\left(V,{}^{-\theta}y\right)$.

This proposition shows that the optimal solution is not affected by changing the state variable. Namely, we get the same solution whether the principal directly maximizes his/her utility given ${}^{-\theta}y$, or first finds the maximum utility he/she can obtain by guaranteeing the agent certain initial utility and then maximizes over the resulting set.

Let $(USCB\left(X,Z\right),\sup)$ denote the space of bounded upper semicontinuous (usc) functions from X to Z endowed with the sup metric. Note that $\left(USCB\left(\left\{V^{ICAP}\left({}^{-\theta}y\right)\right\},\mathbb{R}\right),\sup\right)$ is not a complete metric space. Define $\left\{V^{ICAP}\right\} := \left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}_{{}^{-\theta}y\in Y^{\theta}}$ as the set of possible initial discounted utilities of the agent signing an ICAP contract ordered by initial history. Since Y^{θ} is finite, this set inherits the properties of $\left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$ established in Proposition 4 (a). Then, for $\forall U = \left\{U_{{}^{-\theta}y}\right\}_{{}^{-\theta}y\in Y^{\theta}}$ with $U_{{}^{-\theta}y} \in \left(USCB\left(\left\{V^{ICAP}\left({}^{-\theta}y\right)\right\},\mathbb{R}\right),\sup\right)$, $\forall {}^{-\theta}y \in Y^{\theta}$, define the operator T as follows: $\forall V \in \left\{V^{ICAP}\right\}$, $T\left(U\right)_{(V)} := \left\{T_{{}^{-\theta}y}\left(U\right)_{\left(V_{{}^{-\theta}y}\right)}\right\}_{{}^{-\theta}y\in Y^{\theta}}$ with

$$T_{{}^{-\theta}y}\left(U\right)_{\left(V_{{}^{-\theta}y}\right)} := \max_{c_{{}^{-\theta}y}\left(V_{{}^{-\theta}y}\right)}\Big\{\sum_{y\in Y}[y - w_{{}^{-\theta}y}\left(V_{{}^{-\theta}y},y\right) + \\ +\beta_P U_{{}^{-\theta+1}y,y}\left(V_{+\ {}^{-\theta}y}\left(V_{{}^{-\theta}y},y\right)\right)]\pi\left(y|a_{{}^{-\theta}y}\left(V_{{}^{-\theta}y}\right)\right)\Big\}$$

s.t.

$$a_{{}^{-\theta}y}\left(V_{{}^{-\theta}y}\right) \in A \tag{8}$$

$$w_{{}^{-\theta}y}\left(V_{{}^{-\theta}y},y\right) \in \left[\underline{w},\overline{w}\right],\ \forall y \in Y \tag{9}$$

$$\sum_{y\in Y}[\upsilon\left(w_{-\theta_y}\left(V_{-\theta_y},y\right)\right)-a_{-\theta_y}\left(V_{-\theta_y}\right)+$$

$$+\beta_A V_{+\ -\theta_y}\left(V_{-\theta_y},y\right)]\pi\left(y|a_{-\theta_y}\left(V_{-\theta_y}\right)\right)\geq$$

$$\geq\sum_{y\in Y}[\upsilon\left(w_{-\theta_y}\left(V_{-\theta_y},y\right)\right)-a'_{-\theta_y}\left(V_{-\theta_y}\right)+$$

$$+\beta_A V_{+\ -\theta_y}\left(V_{-\theta_y},y\right)]\pi\left(y|a'_{-\theta_y}\left(V\right)\right),\ \forall a'_{-\theta_y}\left(V_{-\theta_y}\right)\in A \tag{10}$$

$$\sum_{y\in Y}[\upsilon\left(w_{-\theta_y}\left(V_{-\theta_y},y\right)\right)-a_{-\theta_y}\left(V_{-\theta_y}\right)+$$

$$+\beta_A V_{+\ -\theta_y}\left(V_{-\theta_y},y\right)]\pi\left(y|a_{-\theta_y}\left(V_{-\theta_y}\right)\right)=V_{-\theta_y} \tag{11}$$

$$V_{+\ -\theta_y}\left(V_{-\theta_y},y\right)\in\left\{V^{ICAP}\left({}^{-\theta+1}y,y\right)\right\},\ \forall y\in Y \tag{12}$$

Here $V_{+\ -\theta_y}(V,y)$ denotes the agent's expected discounted utility tomorrow given his/her expected discounted utility today (after a history ${}^{-\theta}y$) is V and the stock price realization is y.

The use of max instead of sup in the definition of T is justified by the fact that we are maximizing an usc function over a compact set. Constraints (8), (9), and (10) are the stationary versions of (1), (2′), and (7) respectively. (11) is a promise keeping constraint[7], which guarantees the agent an expected discounted utility of $V_{-\theta_y}$ today. (12) is a consistency constraint requiring that the discounted expected utility that the agent will get next period can be supported by an ICAP supercontract. Note that (12) implies that $V_{+\ -\theta_y}(V,y)\geq \underline{V}_{-\theta+1_{y},y}$, $\forall y\in Y$.

For $\forall{}^{-\theta}y \in Y^{\theta}$ and $\forall V_{-\theta_y} \in \left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$, we have that $\left\{U^{ICAP}\left(V_{-\theta_y},{}^{-\theta}y\right)\right\}$ is compact by Proposition 4(b). Then, we can define $U^{ICAP^*}\left(V_{-\theta_y},{}^{-\theta}y\right):=\max\left\{U^{ICAP}\left(V_{-\theta_y},{}^{-\theta}y\right)\right\}$ as the maximum utility the principal can get by signing an ICAP supercontract offering $V_{-\theta_y}$ to the agent. For $\forall V\in\left\{V^{ICAP}\right\}$, let $U^{ICAP^*}(V)=\left\{U^{ICAP^*}\left(V_{-\theta_y},{}^{-\theta}y\right)\right\}_{{}^{-\theta}y\in Y^{\theta}}$ be the vector of these maximum utilities ordered by initial history. Next, I will show that $U^{ICAP^*}:\left\{V^{ICAP}\right\}\rightarrow\mathbb{R}^{N\theta}$ is the unique fixed point of the operator T and can obtained as the limit of successively iterating on T. I start with a proposition that establishes some useful properties of U^{ICAP^*}.

[7] It is referred to as a re-generation constraint in Spear and Srivastava (1987).

Proposition 7: *$\forall^{-\theta}y \in Y^{\theta}$, $U^{ICAP^*}\left(.,^{-\theta}y\right)$ is usc and bounded on $\left\{V^{ICAP}\left(^{-\theta}y\right)\right\}$.*

Note that these properties can directly be translated to U^{ICAP^*}, say with the sup metric over Y^{θ}.

Proposition 8: $T\left(U^{ICAP^*}\right) = U^{ICAP^*}$.

The proposition says that U^{ICAP^*} is a fixed point of the operator T.

For the purposes of the next proposition, I introduce some additional notation. Let $B(X,Z)$ denote the space of bounded functions from X to Z endowed with the sup metric. For $\forall U', U'' \in \left\{\left(USCB\left(\left\{V^{ICAP}\left(^{-\theta}y\right)\right\}, \mathbb{R}\right), \sup\right)\right\}_{^{-\theta}y \in Y^{\theta}}$, define $\mu\left(U', U''\right) := = \sup\limits_{^{-\theta}y \in Y^{\theta}} \mu_{^{-\theta}y}\left(U', U''\right)$, where $\mu_{^{-\theta}y}\left(U', U''\right) := = \sup\limits_{V_{^{-\theta}y} \in \left\{V^{ICAP}\left(^{-\theta}y\right)\right\}} \left|U'_{^{-\theta}y}\left(V_{^{-\theta}y}\right) - U''_{^{-\theta}y}\left(V_{^{-\theta}y}\right)\right|$, $\forall^{-\theta}y \in Y^{\theta}$. Note that both suprema in the above definition are achieved.

Proposition 9: *(a) T maps $\left(\left\{USCB\left(\left\{V^{ICAP}\left(^{-\theta}y\right)\right\}, \mathbb{R}\right)\right\}_{^{-\theta}y \in Y^{\theta}}, \mu\right)$ into itself; (b) T is a contraction mapping with modulus β_P in terms of the metric μ; (c) Let $\widetilde{U} \in \left(\left\{B\left(\left\{V^{ICAP}\left(^{-\theta}y\right)\right\}, \mathbb{R}\right)\right\}_{^{-\theta}y \in Y^{\theta}}, \mu\right) : T\left(\widetilde{U}\right) = \widetilde{U}$. Then, $\widetilde{U} = U^{ICAP^*}$; (d) $\forall U \in \left(\left\{USCB\left(\left\{V^{ICAP}\left(^{-\theta}y\right)\right\}, \mathbb{R}\right)\right\}_{^{-\theta}y \in Y^{\theta}}, \mu\right)$, $\mu\left(T^n(U), U^{ICAP^*}\right) \underset{n \to \infty}{\to} 0$, where $T^n(U) := \underbrace{T(T(...T}_{n \text{ times}}(U))$ for $\forall n \in \mathbb{Z}_{++}$ with $T^0(U) := U$.*

This proposition shows that the fixed point of T is unique and can be obtained as a limit of successive iterations on T. Consequently, we can use standard dynamic programming techniques in order to solve for the optimal ICAP contract.

However, what we are ultimately interested in is solving for the optimal IC2P contract. For this purpose, I resort again to dynamic programming using a method outlined by Rustichini (1998).

First, I will introduce some notation. For $\forall^{-\theta}y \in Y^{\theta}$ and $V_{^{-\theta}y} \in \left\{V^{ICAP}\left(^{-\theta}y\right)\right\}$, let $\Gamma_R\left(V_{^{-\theta}y}, U,^{-\theta}y\right) := \{c_{^{-\theta}y}\left(V_{^{-\theta}y}\right) : (8) - (12)$ hold after $^{-\theta}y$ and $U_{^{-\theta+1}y,y}\left(V_{+\ ^{-\theta}y}\left(V_{^{-\theta}y}, y\right)\right) \geq \underline{U}_{^{-\theta+1}y,y}$, $\forall y \in Y\}$ for some function $U : \left\{V^{ICAP}\right\} \to (\mathbb{R} \cup \{-\infty\})^{\theta N}$. Additionally, let $\Lambda_R\left(V_{^{-\theta}y}, U,^{-\theta}y\right) := \Gamma_R\left(V_{^{-\theta}y}, U,^{-\theta}y\right)$ if $U_{^{-\theta}y}\left(V_{^{-\theta}y}\right) \geq \underline{U}_{^{-\theta}y}$ and $\Lambda_R\left(V_{^{-\theta}y}, U,^{-\theta}y\right) := \emptyset$ otherwise. Denote by $USCBA\left(\left\{V^{ICAP}\left(^{-\theta}y\right)\right\}, \mathbb{R} \cup \{-\infty\}\right)$ the space of usc, bounded from above functions from X to Z. Then, for $\forall U = \left\{U_{^{-\theta}y}\right\}_{^{-\theta}y \in Y^{\theta}}$ with $U_{^{-\theta}y} \in USCBA\left(\left\{V^{ICAP}\left(^{-\theta}y\right)\right\}, \mathbb{R} \cup \{-\infty\}\right)$, $\forall^{-\theta}y \in Y^{\theta}$, define the operator $\underline{T}$ as follows: $\forall V \in \left\{V^{ICAP}\right\}$, $\underline{T}(U)_{(V)} := \left\{\underline{T}_{-\theta y}(U)_{\left(V_{-\theta y}\right)}\right\}_{-\theta y \in Y^{\theta}}$ where $\underline{T}_{-\theta y}(U)_{\left(V_{-\theta y}\right)} := -\infty$ if $\Lambda_R\left(V_{-\theta y}, U, {}^{-\theta} y\right) = \emptyset$, and $\underline{T}_{-\theta y}(U)_{\left(V_{-\theta y}\right)} := \max\limits_{\substack{c_{-\theta y}\left(V_{-\theta y}\right) \in \\ \Lambda_R\left(V_{-\theta y}, U, {}^{-\theta} y\right)}} \{ \sum\limits_{y \in Y} [y - w_{-\theta y}\left(V_{-\theta y}, y\right) + +\beta_P U_{-\theta+1 y, y}\left(V_{+\ -\theta y}\left(V_{-\theta y}, y\right)\right)] \pi\left(y | a_{-\theta y}\left(V_{-\theta y}\right)\right)\}$ otherwise.

This operator encompasses the lower bounds on the utility of the principal in the form of additional constraints. The only difference with T is that in case U is lower than the reservation utility of the principal today or at any possible contingency tomorrow, $\underline{T}$ becomes $-\infty$. The idea is that any violation of the constraints in this stationary framework is punished severely making the contract in question non-optimal. What remains to be shown is that iterating on this operator will indeed lead us to the optimal dynamic contract.

Proposition 10: *$\underline{T}$ maps $\left\{USCBA\left(\{V^{ICAP}\left({}^{-\theta} y\right)\}, \mathbb{R} \cup \{-\infty\}\right)\right\}_{-\theta y \in Y^{\theta}}$ into itself.*

For $\forall V \in \left\{V^{ICAP}\right\}$, let $D_0(V) := U^{ICAP^*}(V)$ and $D_{i+1}(V) := \underline{T}(D_i)$, $\forall i \in \mathbb{Z}_+$. Note that by Proposition 10 and the fact that $\Gamma_R\left(V_{-\theta y}, U, {}^{-\theta} y\right)$ is compact if non-empty for $\forall V_{-\theta y} \in \left\{V^{ICAP}\left({}^{-\theta} y\right)\right\}$, $\forall U \in \left\{USCBA\left(\{V^{ICAP}\left({}^{-\theta} \widetilde{y}\right)\}, \mathbb{R} \cup \{-\infty\}\right)\right\}_{-\theta \widetilde{y} \in Y^{\theta}}$, $\forall {}^{-\theta} y \in Y^{\theta}$ (trivial), D_i is well defined on V^{ICAP} for $\forall i \in \mathbb{Z}_+$.

Proposition 11: *(a) $\{D_i\}_{i \in \mathbb{Z}_+}$ is a weakly decreasing sequence and $\exists D_{\infty} \in \left\{USCBA\left(\{V^{ICAP}\left({}^{-\theta} y\right)\}, \mathbb{R} \cup \{-\infty\}\right)\right\}_{-\theta y \in Y^{\theta}} : D_i\left(V_{-\theta y}, {}^{-\theta} y\right) \underset{i \to \infty}{\to} D_{\infty}\left(V_{-\theta y}, {}^{-\theta} y\right)$, $\forall V_{-\theta y} \in \left\{V^{ICAP}\left({}^{-\theta} y\right)\right\}$, $\forall {}^{-\theta} y \in Y^{\theta}$; (b) $\underline{T}(D_{\infty}) = D_{\infty}$; and (c) if $\exists D' \in \left\{USCBA\left(\{V^{ICAP}\left({}^{-\theta} y\right)\}, \mathbb{R} \cup \{-\infty\}\right)\right\}_{-\theta y \in Y^{\theta}} : \underline{T}(D') = D'$, then $D' \leq D_{\infty}$.*

This proposition says that if we start iterating on the operator $\underline{T}$ taking U^{ICAP^*} as an initial guess, we will ultimately converge (pointwise) to D_{∞}, the largest fixed point of $\underline{T}$. Next, I establish the relationship between U^* and D_{∞}.

In the subsequent analysis it will be useful to extend U^* on $\{V^{ICAP}\}$. For $\forall V \in \{V^{ICAP}\}$, let $\widehat{U}^*(V) := \left\{\widehat{U}^*\left(V_{-\theta y}, {}^{-\theta} y\right)\right\}_{-\theta y \in Y^{\theta}}$ with $\widehat{U}^*\left(V_{-\theta y}, {}^{-\theta} y\right) = U^*\left(V_{-\theta y}, {}^{-\theta} y\right)$ if $V_{-\theta y} \in \{V^{IC2P}\left({}^{-\theta} y\right)\}$ with $\widehat{U}^*\left(V_{-\theta y}, {}^{-\theta} y\right) := -\infty$ otherwise.

Proposition 12: $\underline{T}\left(\widehat{U}^*\right) = \widehat{U}^*$.

This proposition establishes that the extension of U^* on $\{V^{ICAP}\left({}^{-\theta} y\right)\}$ is a fixed point of $\underline{T}$. What remains to be shown is how to recover U^* from D_{∞}. The next proposition gives the answer.

Proposition 13: For $\forall V \in \{V^{ICAP}\}$, $\widehat{U}^*(V) = D_\infty(V)$.

The proposition provides a straight-forward method of solving for the optimal IC2P supercontract. After we have found the optimal ICAP contract we take it as an initial guess and start iterating on the operator $\underline{T}$ until convergence is reached. Note that convergence here is pointwise and is meant to be on $\mathbb{R} \cup \{-\infty\}$. After we have obtained the limit function D_∞, we can recover the set of possible values for the expected discounted utility of the agent signing an IC2P contract by taking the subset of the domain of D_∞ on which the limit function takes finite values. More precisely, for $\forall {}^{-\theta}y \in Y^\theta$ we can restrict ourselves only to values of $D_\infty\left(.,{}^{-\theta}y\right)$ above $\underline{U}_{{}^{-\theta}y}$. Formally, $\{V^{IC2P}\left({}^{-\theta}y\right)\} := \left\{V \in \{V^{ICAP}\left({}^{-\theta}y\right)\} : D_\infty\left(V,{}^{-\theta}y\right) \geq \underline{U}_{{}^{-\theta}y}\right\}$. Then, for $\forall V \in \{V^{IC2P}\left({}^{-\theta}y\right)\}$, we have $U^*\left(V;{}^{-\theta}y\right) = D_\infty\left(V,{}^{-\theta}y\right)$.

However, note that the state space of the recursive problem constructed for computing the optimal ICAP contract, $\{V^{ICAP}\}$, is endogenous. Nevertheless, it is the largest fixed point of a set operator and can be obtained through successive iterations in a procedure introduced by Abreu, Pearce and Stacchetti (1990).

Choose some $\widehat{V} \in \mathbb{R} : \widehat{V} \geq \max\limits_{{}^{-\theta}y \in Y^\theta}\left\{\max\left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}\right\}$, where the right side of the inequality is well defined given $\left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$ compact, $\forall {}^{-\theta}y \in Y^\theta$ and Y^θ finite. Note that given Assumption 3, $\left[\underline{V}_{{}^{-\theta}y}, \widehat{V}\right] \neq \emptyset$, $\forall {}^{-\theta}y \in Y^\theta$. Then, for $\forall W = \left\{W_{{}^{-\theta}y}\right\}_{{}^{-\theta}y \in Y^\theta} : W_{{}^{-\theta}y} \in \mathbb{R}$, $\forall {}^{-\theta}y \in Y^\theta$ let $B(W) := \left\{B_{{}^{-\theta}y}(W)\right\}_{{}^{-\theta}y \in Y^\theta}$ with $B_{{}^{-\theta}y}(W) := \{V_{{}^{-\theta}y} \in \left[\underline{V}_{{}^{-\theta}y}, \widehat{V}\right] : \exists c_{{}^{-\theta}y}\left(V_{{}^{-\theta}y}\right)$: (8)-(11) and (12′) hold$\}$, where (12′) is defined as $V_{+\,{}^{-\theta}y}\left(V_{{}^{-\theta}y}, y\right) \in W_{{}^{-\theta+1}y,y} \cap \left[\underline{V}_{{}^{-\theta+1}y,y}, +\infty\right), \forall y \in Y$.

Proposition 14: (a) $B\left(\left\{V^{ICAP}\right\}\right) = \left\{V^{ICAP}\right\}$; and (b) *if* $\exists W \subset \mathbb{R}^{N^\theta}$: $B(W) = W$, *then* $W \subset \left\{V^{ICAP}\right\}$.

This proposition establishes that the set of agent's expected discounted utilities supportable by an ICAP supercontract is the largest fixed point of B.

Proposition 15: *Let* W_0 *compact* : $\left\{V^{ICAP}\right\} \subset W_0 \subset \mathbb{R}^{N^\theta}$ *and* $B(W_0) \subset W_0$. *Define* $W_{i+1} := B(W_i)$ *for* $\forall i \in \mathbb{Z}_+$. *Then,* $W_{i+1} \subset W_i$, $\forall i \in \mathbb{Z}_+$ *and* $W_\infty := \lim\limits_{i\to\infty} W_i = \left\{V^{ICAP}\right\}$.

The proposition says that if we start iterating on B taking as an initial guess some compact set W_0 that contains both $B(W_0)$ and $\left\{V^{ICAP}\right\}$, we will ultimately converge to the largest fixed point of the operator, $\left\{V^{ICAP}\right\}$. This is sufficient for obtaining $\left\{V^{ICAP}\right\}$ since we can always take $W_0 = \left\{W_{{}^{-\theta}y}\right\}_{{}^{-\theta}y \in Y^\theta}$: $\left[\underline{V}_{{}^{-\theta}y}, \widehat{V}\right] \subset W_{{}^{-\theta}y} \subset \mathbb{R}$ with $W_{{}^{-\theta}y}$ compact, $\forall {}^{-\theta}y \in Y^\theta$. However, an even more computationally efficient result exists.

Let us modify the operator B as follows. For $\forall W = \left\{W_{-\theta y}\right\}_{-\theta y \in Y^\theta}$: $W_{-\theta y} \in \mathbb{R}$, $\forall^{-\theta} y \in Y^\theta$ let $\widetilde{B}(W) := \left\{\widetilde{B}_{-\theta y}(W)\right\}_{-\theta y \in Y^\theta}$ with $\widetilde{B}_{-\theta y}(W) := \{V_{-\theta y} \in W_{-\theta y} : \exists c_{-\theta y}\left(V_{-\theta y}\right) : \text{(8)-(11) and (12}''\text{) hold}\}$, where (12″) is defined as $V_{+\ -\theta y}\left(V_{-\theta y}, y\right) \in W_{-\theta + 1_{y}, y}$. Note that the operator $\widetilde{B}$ does not require that the agent should commit to the contract. Namely, we do not impose a constraint keeping the continuation values for the utility of the agent above the lower bound given by the reservation utility. From a computational point of view, we are increasing the efficiency since we are relaxing the set of constraints.

Proposition 16: *Take* $\widetilde{W}_0 := \left\{\widetilde{W}_0\left({}^{-\theta}y\right)\right\}_{-\theta y \in Y^\theta}$ *with* $\widetilde{W}_0\left({}^{-\theta}y\right) = \left[\underline{V}_{-\theta y}, \widehat{V}\right]$, $\forall^{-\theta} y \in Y^\theta$ *and let* $\widetilde{W}_{i+1} := \widetilde{B}\left(\widetilde{W}_i\right)$ *for* $\forall i \in \mathbb{Z}_+$. *Then,* $\widetilde{W}_{i+1} \subset \widetilde{W}_i$, $\forall i \in \mathbb{Z}_+$ *and* $\widetilde{W}_\infty := \lim_{i \to \infty} \widetilde{W}_i = \left\{V^{ICAP}\right\}$.

This proposition outlines a practical way of obtaining $\left\{V^{ICAP}\right\}$. Namely, we start with the set $\left\{\left[\underline{V}_{-\theta y}, \widehat{V}\right]\right\}_{-\theta y \in Y^\theta}$ and iterate on the set operator $\widetilde{B}$ until convergence in a properly defined sense is attained. Note that we can always take $\widehat{V} = \frac{v(\overline{w}) - a}{1 - \beta_A}$.

4 Computation and Results

The computation of the model starts with solving for $\left\{V^{ICAP}\right\}$, the set of agent's expected discounted utilities supportable by an ICAP contract. While Proposition 16 gives the theoretical background for the estimation of $\left\{V^{ICAP}\right\}$, some caveats remain. In particular, $\widetilde{B}$ is a set operator and in order to apply the iterative procedure in practice we need an efficient representation of the sequence of sets $\left\{\widetilde{W}_i\right\}_{i \in Z_+}$. For the class of infinitely repeated games with perfect monitoring, Judd, Yeltekin and Conklin (2003) are able to construct inner and outer convex polytope approximations based on the convexification of the equilibrium value set through a public randomization device. Here, I follow a more general approach which does not rely on assuming that $\left\{V^{ICAP}\right\}$ is convex or convexifying it by introducing public randomization.[8] The main idea is to discretize the elements of the initial guess $\widetilde{W}_0$ and start extracting small open intervals, the midpoints of which are unfeasible with respect to $\widetilde{W}_0$. The extraction is done elementwise without updating the previous elements. In particular,

[8]Such a general approach is particularly useful in addressing extensions as for example estimating the endogenous state space of agent's expected discounted utilities supportable by an ICAP stock option contract, because of the non-convexities inherent to the stock option contract.

I start from the discretization of the first[9] element of $\widetilde{W}_0$, find the points that cannot be supported by a one-period ICAP contract with a continuation utility profile contained in $\widetilde{W}_0$, i.e. the points of the discretization which are not in $\widetilde{B}_1\left(\widetilde{W}_0\right)$, and extract small open balls around these points. Next, I find the gridpoints in the second element of $\widetilde{W}_0$ which are unfeasible with respect to $\widetilde{W}_0$, extract their small open neighborhoods and proceed in a similar fashion until I cover all the elements of $\widetilde{W}_0$. The remaining set, i.e. $\widetilde{W}_0$ less the extracted intervals, becomes $\widetilde{W}_1$, our new guess for $\left\{V^{ICAP}\right\}$. Given that $\widetilde{W}_0$ is a vector of N^θ closed intervals in $\mathbb{R}$, each of the N^θ elements of $\widetilde{W}_1$ will be a finite union of closed intervals in $\mathbb{R}$. In order to increase efficiency, intervals with length less than some prespecified level are reduced to their midpoints. The procedure stops if for each element of $\widetilde{W}_i$ the number of closed intervals representing it equals the respective number for the same[10] element in $\widetilde{W}_{i-1}$ and, in addition, the representation of $\widetilde{W}_i$ differs from the representation of $\widetilde{W}_{i-1}$ by less than some prespecified tolerance level. In order to apply this stopping criterion, one still needs to construct a measure for the difference between representations. For this purpose, I find the difference in absolute terms between each endpoint (minimum or maximum point) of each interval of each element of $\widetilde{W}_i$ and $\widetilde{W}_{i-1}$ respectively and take the maximum one to be the difference between the representations of $\widetilde{W}_i$ and $\widetilde{W}_{i-1}$. This difference is well defined given that the two representations share the same structure, which is actually the first condition of the stopping criterion.

Once $\left\{V^{ICAP}\right\}$ is obtained, it is elementwise discretized and used as a state space in the dynamic program for obtaining U^{ICAP^*} as outlined in Proposition 9. At each iteration, the guess for U^{ICAP^*} being defined only on the discretization needs to be interpolated over the state space. Interpolation is also required in the subsequent iterative procedure which uses U^{ICAP^*} as an initial guess for $\widehat{U}^*$, the extension of U^* on $\left\{V^{ICAP}\right\}$.

Table 1 in Appendix 2 contains $\left\{V^{ICAP}\right\}$, the state space of the optimal ICAP contract. The results are obtained by parameterizing the model in line with the calibration of Aseff and Santos (2005). Namely, the set of possible stock prices $Y = \left\{y_{(1)}, y_{(2)}, y_{(3)}\right\} = \{0.55, 1.125, 1.7\}$, the action space $A = \{\underline{a}, \overline{a}\} = \{0.1253, 0.1469\}$, the conditional probabilities $\pi\left(y_{(1)}|\underline{a}\right) = 0.1508$, $\pi\left(y_{(2)}|\underline{a}\right) = 0.8121$, $\pi\left(y_{(3)}|\underline{a}\right) = 0.0371$, $\pi\left(y_{(1)}|\overline{a}\right) = 0.1268$, $\pi\left(y_{(2)}|\overline{a}\right) = 0.8082$, $\pi\left(y_{(3)}|\overline{a}\right) = 0.065$.[11] I fix $\underline{w} = 0$ and equalize the discount factors for the agent and the principal $\beta_A = \beta_P = 0.96$. The period utility with no effort, $\upsilon(.) = \sqrt{(.)}$, is as in Aseff (2004). Furthermore, I consider three different cases.

[9] I endow the set of possible initial histories Y^θ with the lexicographic order and order the elements of $\widetilde{W}_0$ accordingly.

[10] Here, 'same' refers to the index of the element, i.e. to the initial history to which it corresponds.

[11] Aseff and Santos (2005) actually consider two conditional distributions over an interval of possible stock option prices [0.55, 1.7]. In this numerical experiment, I concentrate the mass of each distribution on 3 points of this interval: the minimum, middle, and maximum point.

Case 1 follows the theoretical analysis in Section 2 to derive the uniform upper bound for the wage $\overline{w}$ given the minimum reservation utility of the principal $\underline{U}$. Cases 2 and 3 still honor the upper bound for the wage $\overline{w}$, but impose additional restrictions on the agent's period consumption.[12] Case 2 bounds the wage by $\overline{y}$ at each contingency. It implicitly allows the agent to borrow up to $\overline{y}-y$ units of consumption every period given a current stock price realization y. Case 3 implicitly prevents the agent from borrowing. At each possible contingency, he/she can consume no more than the current stock price realization. For case 1, I take the upper bound for the initial guess $\widehat{V}=\frac{\upsilon(\overline{w})-a}{1-\beta_A}$, where $\overline{w}$ is derived for $\underline{U}=0$. I analyze the case of $\theta=1$, which encompasses $\theta=0$ as a subcase. Then, I have to deal with $N^{\theta}=3$ (initial history) states. I use the natural notation $y_{(i)}$ for the state with initial history $y_{(i)}$, $i\in\{1,2,3\}$. I consider three possible values for the reservation utility of the agent: L= $\frac{\upsilon(\underline{w})-\overline{a}}{1-\beta_A}$ = -3.6725, M = 0, H = -L. Then, I analyze the more interesting case of nonnegative correlation between initial histories and agent's reservation utilities. This limits the number of possible combinations of reservation utility values across initial histories to 10. For example, LMH, which stays for $\underline{V}\left(y_{(1)}\right)$ = L, $\underline{V}\left(y_{(2)}\right)$ = M, $\underline{V}\left(y_{(3)}\right)$ = H, is allowed while LHM is not. Note that KKK is equivalent to the case of $\theta=0$ and $\underline{V}$ =K, where K$\in${L,M,H}. Each cell of Table 1, contains $\left\{V^{ICAP}\right\}$ for a particular combination of reservation utility values (table rows) and a particular case (table columns). In each cell, the left subcolumn corresponds to the intervals' minimum points and the right - to the maximum points, while each subrow corresponds to a particular initial history. For example, for LMH, (case) 1, $\left\{V^{ICAP}\left(y_{(1)}\right)\right\}$ = [0.8275, 843.0178], $\left\{V^{ICAP}\left(y_{(2)}\right)\right\}$ = [0.8200, 843.0178], $\left\{V^{ICAP}\left(y_{(3)}\right)\right\}$ = [3.6725, 843.0178].

The results suggest that for $\forall i\in\{1,2,3\}$, $\left\{V^{ICAP}\left(y_{(i)}\right)\right\}$ is convex from where come the single intervals in Table 1. However, this does not automatically translate to the state space of the stock option contract, which is inherently nonconvex. Note that at least for cases 1 and 2 the upper bound of $\left\{V^{ICAP}\right\}$ remains constant across initial histories and reservation utility combinations. In fact, it equals the theoretical bound given the case: $\frac{\upsilon(\overline{w})-a}{1-\beta_A}$ for case 1 and $\frac{\upsilon(\overline{y})-a}{1-\beta_A}$ for case 2. This means that wages can be high enough to support high expected discounted utilities for the agent. Note, however, that $\left\{V^{IC2P}\right\}\subset\left\{V^{ICAP}\right\}$ and we lose high utility values in solving for U^{*} as Figure 2 in Appendix 2 indicates. The reason is that the value function is decreasing in the upper region of $\left\{V^{ICAP}\right\}$, which results in violations of the principal's participation constraint for high utility values of the agent.

Figure 1 in Appendix 2 plots the value function of the auxiliary problem $U^{ICAP^{*}}(V;.)$ over the endogenous state space $\left\{V^{ICAP}(.)\right\}$ for LMH, case 1. The left panel corresponds to an initial history $y_{(1)}$, the middle - to $y_{(2)}$, and the right - to $y_{(3)}$. While the value function may appear identical across initial histories, this is not true numerically. The illusion comes from the large scale of the graph necessitated by the extensive size of $\left\{V^{ICAP}(.)\right\}$ and the very low

[12] Cf. Wang (1997).

values which $U^{ICAP^*}(V;.)$ takes for V high enough. For this purpose, Figure 2 plots $U^{ICAP^*}(V;.)$ over a small region of $\{V^{ICAP}(.)\}$ containing all the values of $\{V^{IC2P}(.)\}$. The big difference actually comes from the variation in the agent's minimum utility supportable by an ICAP contract. Namely, the higher reservation utility of the agent for initial history $y_{(3)}$ cuts the maximum utility the principal can get by signing the contract. The graph also suggests that the value function is concave and monotonically decreasing. However, it is not easy to generalize these properties. For example, Figure 3 plots $U^{ICAP^*}(V)$ for LLL, case 3. In this particular case, $U^{ICAP^*}(V)$ which actually coincides with $U^*(V)$ fails to be monotonic.

Figure 2 also plots $U^*(V,.)$ over $\{V^{IC2P}(.)\}$. As expected, the maximum utility the principal can get by signing an ICAP contract exceeds the one related to the IC2P contract. Moreover, since the value function in the auxiliary problem is decreasing on most of the domain, the subspace of agent's expected discounted utilities supportable by an IC2P contract is recovered by shrinking $\{V^{ICAP}\}$ from the right.

Regarding the characteristics of the optimal contract, the recommended action is predominantly the high-effort one. However, especially when the initial history is $y_{(3)}$, the low-effort action appears to be optimal in some utility regions. A possible explanation is that the agent is so rich (remember that agent's reservation utility is higher at $y_{(3)}$) that the principal finds motivating him/her to exert effort suboptimal. Figure 4 illustrates the relationship for initial history y_1, LMH, case 3. There, shirking is optimal for sufficiently high current utility values, i.e. when the agent is rich enough. In general, the agent's utility tomorrow increases in the end-of-period stock price realization (see Figure 8 for an illustration for initial history $y_{(1)}$, LMH, case 3). The trend is not so clearly manifested if the initial history is $y_{(3)}$, i.e. if yesterday we observed the highest possible price. If we keep the end-of-period stock price realization fixed, tomorrow's utility increases with the utility today (see, for example, Figure 7). Once again, the trend is least pronounced for initial history $y_{(3)}$. As regards the wage, it increases in current utility and demonstrates a slight upper trend in the end-of-period stock prices (see Figures 5 and 6 respectively).

An interesting observation is that the minimum utility supportable by an ICAP/IC2P contract for initial histories characterized by lower reservation utility is boosted by higher reservation utilities for other states. This suggest that in the presence of positive correlation between stock prices and reservation utilities, the participation constraint of the agent does not bind in states characterized by low stock prices. In other words, the ICAP/IC2P contract provides the agent with some insurance against bad outcomes, which ultimately smooths his/her consumption across (initial history) states. Finally, note that if the reservation utility remains the same across some, but not all of the truncated initial histories, $\{V^{ICAP}(.)\}$ is identical for the initial histories with the same reservation utility. While this seems obvious for $\theta \leq 1$, longer history dependence will potentially break the relation since the set of possible tomorrow's histories will depend on the history today.

Table 2 in Appendix 2 shows the effect of changing the value of the minimum reservation utility of the principal for LLL, case 1. Theoretically, we have that increasing $\underline{U}$ decreases $\overline{w}$, which in turn causes $\widehat{V}$ to fall. Since the analysis so far suggests that the theoretical upper bounds for agent's utility can be supported by an ICAP contract, the only effect of changing $\underline{U}$ comes from the resulting change in the theoretical bound. Moreover, since the IC2P contract cannot support agent's utilities in the upper region of $\left\{V^{ICAP}\left(.\right)\right\}$, the optimal self-enforcing contract is not affected.

Figure 9 in Appendix 2 illustrates the development in time of the expected discounted utility of the agent under the optimal contract given an initial history $y_{(1)}$, LMH, case 3. Each curve on the graph starts from a point $V_0 \in \left\{V^{ICAP}\left(y_{(1)}\right)\right\}$ and represents a "typical"[13] path $V^{100}\left(a^*_{y_{(1)}}, w^*_{y_{(1)}}, y_{(1)}\right)$. The results suggest that in the long run the agent does not get poorer in utility terms. In particular, CEOs who start rich tend to keep their utility level while those who start poor get richer in time.

[13] Each depicted path is taken to be the mean of 100 independently generated paths which are constructed following the transition and the policies of the optimal contract given the initial condition $\left(y_{(1)}, V_0\right)$.

5 Conclusion

This paper builds a framework for analyzing dynamic moral hazard problems characterized by limited commitment and history-dependent reservation utilities. I parameterize the model and compute the optimal contract under different structural arrangements. I find evidence that in the presence of positive correlation between stock prices and reservation utilities, the contract provides the agent with insurance against bad outcomes, which ultimately smooths his/her consumption across (initial history) states. Exerting effort appears to be the predominant strategy for the principal, but shirking may still be optimal when the agent is rich enough. The optimal wage scheme and the future utility of the agent tend to grow in both his/her current utility and in the future realization of the stock price. The agent's utility weakly increases in the long run.

References

[1] Abreu, D., Pearce, D. and E. Stacchetti (1990): "Toward a Theory of Discounted Repeated Games with Imperfect Monitoring," Econometrica, 58, 1041-1063.

[2] Aseff, J. G. (2004): "Numerical Analysis of a Dynamic model of Agency," Mimeo, DePaul University.

[3] Aseff, J. G. and M. S. Santos (2005): "Stock Options and Managerial Optimal Contracts," Economic Theory, 26 (4), 813-837.

[4] Doepke, M. and R. Townsend (2004): "Dynamic Mechanism Design with Hidden Income and Hidden Auctions," C.E.P.R. Discussion Paper 4455.

[5] Fernandes, A. and C. Phelan (2000): "A Recursive Formulation for Repeated Agency with History Dependence," Journal of Economic Theory, 91, 223-247.

[6] Green, E. J. (1987): "Lending and the Smoothing of Uninsurable Income," in Contractual Agreements for Intertemporal Trade, ed. by E. C. Prescott and N. Wallace, Minneapolis: University of Minnesota Press, 3-25.

[7] Grossman, S. J. and O. D. Hart (1983): "An Analysis of the Principal-Agent Problem," Econometrica, 51 (1), 7-46.

[8] Jarque, A. (2005): "Repeated Moral Hazard with Effort Persistence," Mimeo, Universidad de Alicante.

[9] Judd, K. L., Yeltekin, S. and J. Conklin (2003): "Computing Supergame Equilibria," Econometrica, 71 (4), 1239-1254.

[10] Ligon, E., Thomas, J. and T. Worrall (2000) "Mutual Insurance, Individual Savings, and Limited Commitment," Review of Economic Dynamics, 3, 216-246.

[11] Ligon, E., Thomas, J. and T. Worrall (2002) "Informal Insurance Arrangements with Limited Commitment: Theory and Evidence from Village Economies," Review of Economic Studies, 69 (1), 209-244.

[12] Marcet, A and R. Marimon (1998): "Recursive contracts," Universitat Pompeu Fabra Working Paper Ref. 337.

[13] Oyer, P. (2004): "Why Do Firms Use Incentives That Have No Incentive Effects," Journal of Finance, 59 (4), 1619-1649.

[14] Phelan, C. (1995): "Repeated Moral Hazard and One-Sided Commitment," Journal of Economic Theory, 66, 488-506.

[15] Phelan, C. and E. Stacchetti (2001): "Sequential Equilibria in a Ramsey Tax Model," Econometrica, 69 (6), 1491-1518.

[16] Ray, D. (2002): "The Time Structure of Self-Enforcing Agreements," Econometrica, 70 (2), 547-582.

[17] Rustichini, A. (1998): "Dynamic Programming Solution of Incentive Constrained Problems," Journal of Economic Theory, 78, 329-354.

[18] Sleet, C. and S. Yeltekin (2001): "Dynamic Labor Contracts with Temporary Layoffs and Permanent Separations," Economic Theory, 18, 207-235.

[19] Spear, S. E. and S. Srivastava (1987): "On Repeated Moral Hazard with Discounting," Review of Economic Studies, 54 (4), 599-617.

[20] Stockey, N. L. and R. E. Lucas (1989): Recursive Methods in Economic Dynamics, Cambridge: Harvard University Press.

[21] Thomas, J. and T. Worrall (2002) "Unemployment Insurance under Moral Hazard and Limited Commitment: Public vs Private Provision," Public Economics 0211002, EconWPA.

[22] Wang, C. (1997): "Incentives, CEO Compensation, and Shareholder Wealth in a Dynamic Agency Model," Journal of Economic Theory, 76, 72-105.

[23] Wang, C. (2000): "Renegotiation-Proof Dynamic Contracts with Private Information," Review of Economic Dynamics, 3, 396-422.

APPENDIX 1

Proof of Proposition 1: It is trivial to show (6)$\Rightarrow$(3). Just take $\tau = 0$. In the other direction, let (3) hold, but assume that (6) is not satisfied, i.e. $\exists \tau \in \mathbb{Z}_+$ and $\widetilde{y}^{\tau-1} \in Y^\tau$ s.t. $\exists a'_{-\theta_y} : a'_t\left({}^{-\theta}y^{t-1}\right) \in A$, $\forall {}^{-\theta}y^{t-1} \in \left({}^{-\theta}y, \widetilde{y}^{\tau-1}\right) \times Y^{t-\tau}$, $\forall t \in \mathbb{Z}_+ : t \geq \tau$ and $V_\tau\left(a'_{-\theta_y}, w_{-\theta_y}, {}^{-\theta}\widetilde{y}^{\tau-1}\right) > V_\tau\left(a_{-\theta_y}, w_{-\theta_y}, {}^{-\theta}\widetilde{y}^{\tau-1}\right)$. By Assumption 1, for $\forall \tau \in \mathbb{Z}_+$, $\forall {}^{-\theta}y^{\tau-1} \in {}^{-\theta}y \times Y^\tau$, we have that $V_\tau\left(w, \widetilde{a}, y^{\tau-1}\right) = = \sum_{t=\tau}^{\infty} \beta_A{}^{t-\tau} \sum_{y_t \in Y} \ldots \sum_{y_\tau \in Y} \left[\upsilon\left(w_t\left({}^{-\theta}y^t\right)\right) - \widetilde{a}_t\left({}^{-\theta}y^{t-1}\right)\right] \prod_{i=\tau}^{t} \pi\left(y_i | \widetilde{a}_i\left({}^{-\theta}y^{i-1}\right)\right)$ does not depend on $\left\{\widetilde{a}_t\left({}^{-\theta}y^{t-1}\right) : {}^{-\theta}y^{t-1} \in {}^{-\theta}y \times Y^{t-1}\right\}_{t=0}^{\tau-1}$. Let $a''_{-\theta_y}$: $a''_t\left({}^{-\theta}y^{t-1}\right) = a'_t\left({}^{-\theta}y^{t-1}\right)$, $\forall {}^{-\theta}y^{t-1} \in \left({}^{-\theta}y, \widetilde{y}^{\tau-1}\right) \times Y^t$, $\forall t \in \mathbb{Z}_+ : t \geq \tau$ with $a''_t\left({}^{-\theta}y^{t-1}\right) = a_t\left({}^{-\theta}y^{t-1}\right)$ elsewhere. We have $V_\tau\left(a''_{-\theta_y}, w_{-\theta_y}, {}^{-\theta}\widetilde{y}^{\tau-1}\right) = V_\tau\left(a'_{-\theta_y}, w_{-\theta_y}, {}^{-\theta}\widetilde{y}^{\tau-1}\right)$. Using the boundedness of $\upsilon(.)$ and $\pi(.)$ (given (1), (2) and $w(.) \leq \overline{w}$ at every node), we obtain:

$$
\begin{aligned}
&V_0\left(a''_{-\theta_y}, w_{-\theta_y}, {}^{-\theta}y\right) = \\
&= \sum_{t=0}^{\tau-1} \beta_A^t \sum_{y_t \in Y} \ldots \sum_{y_0 \in Y} \left[\upsilon\left(w_t\left({}^{-\theta}y^t\right)\right) - a''_t\left({}^{-\theta}y^{t-1}\right)\right] \prod_{i=0}^{t} \pi\left(y_i | a''_i\left({}^{-\theta}y^{i-1}\right)\right) + \\
&+ \beta_A^\tau \sum_{y_{t-1} \in Y} \ldots \sum_{y_0 \in Y} V_\tau\left(a''_{-\theta_y}, w_{-\theta_y}, {}^{-\theta}y^{\tau-1}\right) \prod_{i=0}^{t-1} \pi\left(y_i | a''_i\left({}^{-\theta}y^{i-1}\right)\right) \\
&\qquad\qquad > V_0\left(a_{-\theta_y}, w_{-\theta_y}, {}^{-\theta}y\right), \qquad \text{(A 1)}
\end{aligned}
$$

where the inequality follows from the construction of $a''_{-\theta_y}$, $V_\tau\left(a'_{-\theta_y}, w_{-\theta_y}, {}^{-\theta}\widetilde{y}^{\tau-1}\right) > V_\tau\left(a_{-\theta_y}, w_{-\theta_y}, {}^{-\theta}\widetilde{y}^{\tau-1}\right)$ and $\pi > 0$ by Assumption 1. Given that $a''_{-\theta_y}$ is admissible after ${}^{-\theta}y$ by construction, (A 1) contradicts (3).$\blacksquare$

Proof of Proposition 2: In order to show that (3) implies (7), just note that by Proposition 1. (3)$\Rightarrow$(6), and that (6)$\Rightarrow$(7). In the other direction, assume (7) holds at every node, but $\exists$ an admissible plan $a'_{-\infty_y}$: $V_0\left(a'_{-\theta_y}, w_{-\theta_y}, {}^{-\theta}y\right) > V_0\left(a'_{-\theta_y}, w_{-\theta_y}, {}^{-\theta}y\right)$ .We have $V_0\left(a'_{-\infty_y}, w_{-\infty_y}, y^{-1}\right) = = \sum_{t=0}^{T} \beta_A^t \sum_{y_t \in Y} \ldots \sum_{y_0 \in Y} [\upsilon\left(w_t\left({}^{-\theta}y^t\right)\right) - a'_t\left({}^{-\theta}y^{t-1}\right)] \prod_{i=0}^{t} \pi\left(y_i | a'_i\left({}^{-\theta}y^{i-1}\right)\right) + + \beta_A^{T+1} \sum_{y_T \in Y} \ldots \sum_{y_0 \in Y} V_{T+1}\left(a'_{-\infty_y}, w_{-\infty_y}, y^T\right) \prod_{i=0}^{T} \pi\left(y_i | a'_i\left({}^{-\theta}y^{i-1}\right)\right)$. The second term on the right-hand side can be made arbitrarily small by choosing T big enough (given the assumptions on β_A, υ, A; (1), (2) and $w(.) \leq \overline{w}$ at every node). Therefore, $\exists T \in \mathbb{Z}_+$ and an admissible plan $a''_{-\theta y} : a''_t\left({}^{-\theta}y^{t-1}\right) = a'_t\left({}^{-\theta}y^{t-1}\right)$, $\forall {}^{-\theta}y^{t-1} \in {}^{-\theta}y \times Y^t$, $\forall t \in \mathbb{Z}_+ : t \leq T$, $a''_t\left({}^{-\theta}y^{t-1}\right) = a_t\left({}^{-\theta}y^{t-1}\right)$, $\forall {}^{-\theta}y^{t-1} \in {}^{-\theta}y \times Y^t$, $\forall t \in \mathbb{Z}_{++} : t > T$ and $V_0\left(a''_{-\theta y}, w_{-\theta y}, {}^{-\theta}y\right) > V_0\left(a_{-\theta y}, w_{-\theta y}, {}^{-\theta}y\right)$ Then, take $\tau \in \mathbb{Z}_+ : \tau \leq T$ s.t. $\exists {}^{-\theta}y^{\tau-1} \in {}^{-\theta}y \times Y^\tau$: $a''_\tau\left({}^{-\theta}y^{\tau-1}\right) \neq a_\tau\left({}^{-\theta}y^{\tau-1}\right)$ and $\nexists \tau' \in \mathbb{Z}_{++} : \tau < \tau' \leq T : a''_{\tau'}\left({}^{-\theta}y^{\tau'-1}\right) \neq a_{\tau'}\left({}^{-\theta}y^{\tau'-1}\right)$ for some ${}^{-\theta}y^{\tau'-1} \in {}^{-\theta}y \times Y^{\tau'}$. If we define an admissible plan $a^*_{-\theta y} : a^*_t\left({}^{-\theta}y^{t-1}\right) = a''_t\left({}^{-\theta}y^{t-1}\right)$, $\forall {}^{-\theta}y^{t-1} \in {}^{-\theta}y \times Y^t$, $\forall t \in \mathbb{Z}_+ \setminus \{\tau\}$, and $a^*_\tau\left({}^{-\theta}y^{\tau-1}\right) = a_\tau\left({}^{-\theta}y^{\tau-1}\right)$, $\forall {}^{-\theta}y^{\tau-1} \in {}^{-\theta}y \times Y^\tau$, by (7) at $\forall {}^{-\theta}y^{\tau-1} \in {}^{-\theta}y \times Y^\tau$, we have that $V_\tau\left(a^*_{-\theta y}, w_{-\theta y}, {}^{-\theta}y^{\tau-1}\right) > V_\tau\left(a''_{-\theta y}, w_{-\theta y}, {}^{-\theta}y^{\tau-1}\right)$, $\forall {}^{-\theta}y^{\tau-1} \in {}^{-\theta}y \times Y^\tau$ from where $V_0\left(a^*_{-\theta y}, w_{-\theta y}, {}^{-\theta}y\right) > V_0\left(a''_{-\theta y}, w_{-\theta y}, {}^{-\theta}y\right)$. Proceeding in this way we can eliminate all the deviations (note that $\tau \in \mathbb{Z}_+ : \tau \leq T$) to get that $V_0\left(a_{-\theta y}, w_{-\theta y}, {}^{-\theta}y\right) > V_0\left(a''_{-\theta y}, w_{-\theta y}, {}^{-\theta}y\right)$, i.e. a contradiction.■

Proof of Proposition 3: Fix $\tau \in \mathbb{Z}_+$ and ${}^{-\theta}y^{\tau-1} \in {}^{-\theta}y \times Y^\tau$. We have $\frac{\overline{y}-\underline{w}}{1-\beta_P} \geq U_\tau\left(a, w, {}^{-\theta}y^{\tau-1}\right) \geq \underline{U}$, where the first inequality follows from (1), (2) and the properties of A, Y, π, β_P and the second - from (5). Therefore, $U_\tau\left(a, w, {}^{-\theta}y^{\tau-1}\right)$ is bounded. Since $\beta_P \in (0,1)$ and $y_t \in \left[\underline{y}, \overline{y}\right]$, where $\underline{y}$, $\overline{y} \in \mathbb{R}_+$ we have that $\widehat{Y}_{-\theta y^{\tau-1}} := \sum_{t=\tau}^{\infty} \beta_P^{t-\tau} \sum_{y_t \in Y} ... \sum_{y_\tau \in Y} y_t \prod_{i=\tau}^{t} \pi\left(y_i | a_i\left({}^{-\theta}y^{i-1}\right)\right) \in \left[\frac{\underline{y}}{1-\beta_P}, \frac{\overline{y}}{1-\beta_P}\right]$. Then, $\widehat{W}\left(a, w, {}^{-\theta}y^{\tau-1}\right) := \sum_{t=\tau}^{\infty} \beta_P^{t-\tau} \sum_{y_t \in Y} ... \sum_{y_\tau \in Y} w_t\left({}^{-\theta}y^t\right) \prod_{i=\tau}^{t} \pi\left(y_i | a_i\left({}^{-\theta}y^{i-1}\right)\right) \in \left[\frac{\underline{w}}{1-\beta_P}, \frac{\overline{y}}{1-\beta_P} - \underline{U}\right]$. Choose an arbitrary node ${}^{-\theta}y'^{\tau} \in {}^{-\theta}y^{\tau-1} \times Y$. Let's fix a for a moment. Since $\widehat{W}\left(a, ., {}^{-\theta}y^{\tau-1}\right)$ is strictly increasing in $w_t\left({}^{-\theta}y^t\right)$, $\forall {}^{-\theta}y^t \in \left({}^{-\theta}y, y^{\tau-1}\right) \times Y^{t-\tau+1}$, $\forall t \in \mathbb{Z}_+ : t \geq \tau$ and bounded from above by $\frac{\overline{y}}{1-\beta_P} - \underline{U}$, we can find $\overline{w}\left({}^{-\theta}y'^{\tau}\right) := \sup_{(w,a):(1),(2),(5) \text{ hold after } {}^{-\theta}y} \left\{w_t\left({}^{-\theta}y'^{\tau}\right)\right\}$ by solving $\widehat{W}\left(a, w_H, {}^{-\theta}y^{\tau-1}\right) = \frac{\overline{y}}{1-\beta_P} - \underline{U}$ for $w_H\left({}^{-\theta}y'^{\tau}\right)$, where $w_H : w_H\left({}^{-\theta}y^t\right) = \underline{w}$, $\forall {}^{-\theta}y^t \in \left\{\left({}^{-\theta}y, y^{\tau-1}\right) \times Y^{t-\tau+1}\right\} \setminus \left\{{}^{-\theta}y'^{\tau}\right\}$, $\forall t \in \mathbb{Z}_+ : t \geq \tau$. Then, we have $\frac{\underline{w}}{1-\beta_P} + \left(\overline{w}\left({}^{-\theta}y'^{\tau}\right) - \underline{w}\right) \pi\left(y'_\tau | a_\tau\left({}^{-\theta}y^{\tau-1}\right)\right) = \frac{\overline{y}}{1-\beta_P} - \underline{U}$, from where $\overline{w}\left({}^{-\theta}y'^{\tau}\right) = \frac{1}{\pi\left(y'_\tau | a_\tau\left({}^{-\theta}y^{\tau-1}\right)\right)}\left(\frac{\overline{y}-\underline{w}}{1-\beta_P} - \underline{U}\right) + \underline{w}$. Note that $\left(\frac{\overline{y}-\underline{w}}{1-\beta_P} - \underline{U}\right)$ is non-negative by Assumption 3. Since $\forall y \in Y$, $\pi(y|.)$ is continuous in A , compact, we have that $\forall y \in Y$, $\exists \widetilde{a}_y \in A : \underline{\pi}(y) := \min_{a_y \in A} \pi(y|a_y) = \pi(y|\widetilde{a}_y)$. Moreover, Y finite, therefore $\exists \widetilde{y} \in Y : \underline{\pi} := \min_{y \in Y} \underline{\pi}(y) = \underline{\pi}(\widetilde{y})$. Then, we have $\underline{\pi} = \pi(\widetilde{y}|\widetilde{a}_{\widetilde{y}}) > 0$, where the inequality follows from Assumption 1. By construction, $\pi\left(y'_\tau|a_\tau\left({}^{-\theta}y^{\tau-1}\right)\right)\geq\underline{\pi}$, therefore $\overline{w}\left({}^{-\theta}y'^{\tau}\right)\leq\frac{1}{\underline{\pi}}\left(\frac{\overline{y}-w}{1-\beta_P}-\underline{U}\right)+\underline{w}$. Since ${}^{-\theta}y^{\tau-1}$ was taken randomly, we are done.■

For $\forall{}^{-\theta}y\in Y^\theta$, let $\Omega_{{}^{-\theta}y}:=\{(a,w):(1)\text{ and }(2')\text{ hold after }{}^{-\theta}y\}$.

Proof of Proposition 4: (a) Fix ${}^{-\theta}y\in Y^\theta$. Take $\tau',\tau''\in\mathbb{Z}_+:\tau'\leq\tau''$ and arbitrary ${}^{-\theta}y'^{\tau'-1}\in Y^{\tau'}\times{}^{-\theta}y$ and ${}^{-\theta}y''^{\tau''-1}\in Y^{\tau''}\times{}^{-\theta}y$. Take an arbitrary $V'\in\left\{V_{\tau'}^{IC2P}\left({}^{-\theta}y'^{\tau'-1}\right)\right\}$. Then $\exists$ a contract $c'=(a',w')$ s.t. (1),(2),(4),(5),(7) hold after ${}^{-\theta}y'^{\tau'-1}$ and $V_{\tau'}\left(c',{}^{-\theta}y'^{\tau'-1}\right)=V'$. Define $c''=(a'',w'')$ s.t. for $\forall{}^{-\theta}\widehat{y}^t\in{}^{-\theta}y''^{\tau''-1}\times Y^{t-\tau''+1}$, $\forall t\in\mathbb{Z}_+:t\geq\tau''$, $a''_t\left({}^{-\theta}\widehat{y}^{t-1}\right):=a'_{\tau'+t-\tau''}\left({}^{-\theta}y'^{\tau'-1+t-\tau''}\right)$, $w''_t\left({}^{-\theta}\widehat{y}^t\right):=w'_{\tau'+t-\tau''}\left({}^{-\theta}y'^{\tau'+t-\tau''}\right)$, where ${}^{-\theta}y'^{\tau'+t-\tau''}\in{}^{-\theta}y^{\tau'-1}\times Y^{t-\tau''+1}:y'_{\tau'+i}=\widehat{y}_{\tau''+i}$, $\forall i\in\mathbb{Z}_+:i\leq t-\tau''$. By Assumption 1 and Proposition 3, $V_{\tau''}\left(c'',{}^{-\theta}y^{\tau''-1}\right)-V'=$

$$=-\lim_{T\to\infty}\beta_A^{T-\tau'}\sum_{n=\tau}^{\tau''-\tau'-1}\sum_{y_{T-n}\in Y}\dots\sum_{y_{\tau'}\in Y}\left[\upsilon\left(w_t\left({}^{-\theta}y^t\right)\right)-a'_t\left({}^{-\theta}y^{t-1}\right)\right]\prod_{i=\tau'}^{T-n}\pi\left(y_i|a'_i\left(y^{i-1}\right)\right)=0$$

and it is straightforward to show that (1),(2),(4),(5),(7) hold after ${}^{-\theta}y^{\tau''-1}$ given the definition of c''. Therefore, we have $V'\in\left\{V_{\tau''}^{IC2P}\left({}^{-\theta}y^{\tau''-1}\right)\right\}$. The same argument holds in the other direction, so we have proven that $\left\{V_{\tau''}^{IC2P}\left({}^{-\theta}y^{\tau'-1}\right)\right\}=\left\{V_{\tau''}^{IC2P}\left({}^{-\theta}y^{\tau''-1}\right)\right\}$.

Fix ${}^{-\theta}y\in Y^\theta$. $\left\{V^{IC2P}\left({}^{-\theta}y\right)\right\}$ is bounded given (1), (2) and (5) hold. Regarding the compactness of $\left\{V^{IC2P}\left({}^{-\theta}y\right)\right\}$, we should also prove that it is closed. Let's take an arbitrary convergent sequence $\{V_i\}_{i=1}^\infty$: $V_i\in\left\{V^{IC2P}\left({}^{-\theta}y\right)\right\}$, $\forall i\in\mathbb{Z}_{++}$ with limit V_∞. We need to show that $V_\infty\in\left\{V^{IC2P}\left({}^{-\theta}y\right)\right\}$. By the construction of the sequence, for $\forall i\in\mathbb{Z}_{++}$, $\exists c_i$: $(1),(2),(4),(5),(7)$ hold after ${}^{-\theta}y$ and $V_0(c_i,{}^{-\theta}y)=V_i$. Moreover, by Proposition 3, $(2')$ holds after ${}^{-\theta}y$ for $\forall i\in\mathbb{Z}_{++}$. Then, for $\forall i\in\mathbb{Z}_{++}$, $c_i\in\Omega_{{}^{-\theta}y}$. Let's endow $\Omega_{{}^{-\theta}y}$ with the product topology. Then, $\Omega_{{}^{-\theta}y}$ is compact as a product of compact spaces. Therefore, $\exists$ a convergent subsequence $\{c_{i_k}\}_{k=1}^\infty$ of $\{c_i\}_{i=1}^\infty$ s.t. $c_\infty:=\lim_{k\to\infty}(c_{i_k})\in\Omega_{{}^{-\theta}y}$. Consequently, c_∞ satisfies (1) and (2) after ${}^{-\theta}y$. For $\forall T\in\mathbb{Z}_+:T\geq\tau$, let $V_\tau^T(c,{}^{-\theta}y^{\tau-1}):=$

$$=\sum_{t=\tau}^{T}\beta_A^{t-\tau}\sum_{y_t\in Y}\dots\sum_{y_\tau\in Y}\left[\upsilon\left(w_t\left({}^{-\theta}y^t\right)\right)-a_t\left({}^{-\theta}y^{t-1}\right)\right]\prod_{i=\tau}^{t}\pi\left(y_i|a_i\left({}^{-\theta}y^{i-1}\right)\right).$$

Note that $V_\tau(c,{}^{-\theta}y^{\tau-1})-V_\tau^T(c,{}^{-\theta}y^{\tau-1})=$

$$=\beta_A^{T+1}\sum_{y_T\in Y}\dots\sum_{y_\tau\in Y}V_{T+1}\left(c,{}^{-\theta}y^{T-1}\right)\prod_{i=\tau}^{T}\pi\left(y_i|a_i\left({}^{-\theta}y^{i-1}\right)\right)\in$$

$[\beta_A^{T+1}\frac{\upsilon(\underline{w})-\overline{a}}{1-\beta_P},\beta_A^{T+1}\frac{\upsilon(\overline{w})-\underline{a}}{1-\beta_P}]$, $\forall T\in\mathbb{Z}_+:T\geq\tau$, $\forall c\in\Omega_{{}^{-\theta}y}$, $\forall{}^{-\theta}y^{\tau-1}\in{}^{-\theta}y\times Y^\tau$, $\forall\tau\in\mathbb{Z}_+$. Moreover, $V_\tau^T(.,{}^{-\theta}y^{\tau-1})$ is continuous on $\Omega_{{}^{-\theta}y}$. Then, $V_\tau(.,{}^{-\theta}y^{\tau-1})$

is continuous on $\Omega_{-\theta_y}$. Analogously, we can show that $U_\tau(.,{}^{-\theta}y^{\tau-1})$ is continuous on $\Omega_{-\theta_y}$. As a result, we have that c_∞ satisfies $(4),(5),(7)$ after $y^{-\theta}$ and $V_0\left(c_\infty,{}^{-\theta}y\right)=V_\infty$.

Following the same logic, we can show that $\left\{V_\tau^{ICAP}\left({}^{-\theta}y^{\tau-1}\right)\right\}$ is time invariant and compact and depends only on the last θ stock price observations prior to signing.

(b) Analogous to the proof of (a).■

Proof of Proposition 5: Fix ${}^{-\theta}y \in Y^\theta$. By Propositions 2 and 3, $\Gamma_{-\theta_y} = \{c:(1),(2),(4),(5),(7) \text{ hold}\} \subset \Omega_{-\theta_y}$. Let's endow $\Omega_{-\theta_y}$ with a metric inducing the product topology. Then, following the argument of the proof of Proposition 4, we obtain that $\Gamma_{-\theta_y}$ is compact and $U_0\left(.,{}^{-\theta}y\right)$ is continuous on $\Omega_{-\theta_y}$.■

Proof of Proposition 6: Fix ${}^{-\theta}y \in Y^\theta$. By Proposition 5, we have that $\exists\left(a^{**},w^{**}\right):(1),(2),(3),(4),(5)$ hold after ${}^{-\theta}y$ and $U_0\left(a^{**},w^{**},{}^{-\theta}y\right)=U^{**}_{-\theta_y}$. Let $V^{**}:=V_0\left(a^{**},w^{**},{}^{-\theta}y\right)$. By Proposition 2, $V^{**}\in\left\{V^{IC2P}\left(\underline{V},\underline{U}\right)\right\}$ and $U^{**}_{-\theta_y}\in\left\{U\left(V^{**};{}^{-\theta}y\right)\right\}$. Therefore, $U^{*}_{-\theta_y}\geq U^{**}_{-\theta_y}$. Suppose $U^{*}_{-\theta_y}>U^{**}_{-\theta_y}$. Then, $\exists V^{*}\in\left\{V^{IC2P}\left({}^{-\theta}y\right)\right\}:U^{**}_{-\theta_y}<U^{*}\left(V^{*},{}^{-\theta}y\right)\leq U^{*}_{-\theta_y}$. Since $U^{*}\left(V^{*},{}^{-\theta}y\right)\in\left\{U\left(V^{*},{}^{-\theta}y\right)\right\}$, $\exists\left(a^{*},w^{*}\right)\mid(1),(2),(4),(5),(7)$ hold after ${}^{-\theta}y$, $V_0\left(a^{*},w^{*},{}^{-\theta}y\right)=V^{*}$ and $U_0\left(a^{*},w^{*},{}^{-\theta}y\right)=U^{*}\left(V^{*},{}^{-\theta}y\right)$. Then, by the definition of $U^{**}_{-\theta_y}$ and Proposition 2 we have that $U^{**}_{-\theta_y}\geq U^{*}\left(V^{*},{}^{-\theta}y\right)$, i.e. a contradiction is reached. Consequently, $U^{*}_{-\theta_y}=U^{**}_{-\theta_y}$ and the supremum in the definition of $U^{*}_{-\theta_y}$ is achieved.■

For $\forall{}^{-\theta}y\in Y^\theta$ and $\forall V\in\left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$, define $\Gamma^{ICAP}_{-\theta_y}(V):=\Big\{c:(1),(2'),(4),(7)\text{ hold after }{}^{-\theta}y\text{ and }V_0\left(a,w,{}^{-\theta}y\right)=V\Big\}$ and $G^{ICAP}_{-\theta_y}(V):=\Big\{c\in\Gamma^{ICAP}_{-\theta_y}(V):U_0\left(c,{}^{-\theta}y\right)=U^{ICAP^*}\left(V,{}^{-\theta}y\right)\Big\}$.

Lemma 1: *For $\forall{}^{-\theta}y\in Y^\theta$, $\Gamma^{ICAP}_{-\theta_y}(.)$ is upper hemi-continuous (uhc) on $\left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$.*

Proof: Fix ${}^{-\theta}y\in Y^\theta$ and $V\in\left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$ and note that $\Gamma^{ICAP}_{-\theta_y}(V)$ is non-empty and compact. Take a sequence $\{V_i\}_{i=1}^\infty$ s.t. $V_i\in\left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$, $\forall i\in\mathbb{Z}_{++}$ and $V_i\underset{i\to\infty}{\to}V$. Let $c_i\in\Gamma^{ICAP}_{-\theta_y}(V_i)$ for $\forall i\in\mathbb{Z}_{++}$. Note that $\Gamma^{ICAP}_{-\theta_y}(V_i)\subset\Omega_{-\theta_y}$, $\forall i\in\mathbb{Z}_{++}$ with $\Omega_{-\theta_y}$ compact . Then, $\exists$ a subsequence $\left\{c_{i_j}\right\}_{j=1}^\infty$ of $\{c_i\}_{i=1}^\infty:c_{i_j}\underset{j\to\infty}{\to}c\in\Omega_{-\theta_y}$. Since $V_\tau\left(.,{}^{-\theta}y^{\tau-1}\right)$ is continuous on $\Omega_{-\theta_y}$, c satisfies (4) and (7) after ${}^{-\theta}y$ and $V_0\left(c,{}^{-\theta}y\right)=V$. Therefore, $c\in\Gamma^{ICAP}_{-\theta_y}(V)$.■

Proof of Proposition 7: Fix ${}^{-\theta}y\in Y^\theta$ and $V\in\left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$. Take a sequence $\{V_i\}_{i=1}^\infty$ s.t. $V_i\in\left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$, $\forall i\in\mathbb{Z}_{++}$ and $V_i\underset{i\to\infty}{\to}V$. Let

$c_i \in G^{ICAP}_{-\theta y}(V_i)$ for $\forall i \in \mathbb{Z}_{++}$. Define $\overline{U^{ICAP^*}_{-\theta y}} := \overline{\lim}_{V_i \to V} U^{ICAP^*}\left(V_i, {}^{-\theta}y\right)$. $\exists$ a subsequence $\left\{c_{i_j}\right\}_{j=1}^{\infty}$ of $\{c_i\}_{i=1}^{\infty}$: $\lim_{j\to\infty} U_0\left(c_{i_j}, {}^{-\theta}y\right) = \overline{U^{ICAP^*}_{-\theta y}}$. Since $G^{ICAP}_{-\theta y}(.) \subset \Gamma^{ICAP}_{-\theta y}(.)$ and $\Gamma^{ICAP}_{-\theta y}(.)$ is uhc from Lemma 1, $\exists$ a subsequence $\left\{c_{i_{j_n}}\right\}_{n=1}^{\infty}$ of $\left\{c_{i_j}\right\}_{j=1}^{\infty}$: $c_{i_{j_n}} \underset{n\to\infty}{\to} c$ with $c \in \Gamma^{ICAP}_{-\theta y}(V)$. Then, $\overline{U^{ICAP^*}_{-\theta y}} = \lim_{n\to\infty} U_0\left(c_{i_{j_n}}, {}^{-\theta}y\right) = U_0\left(c, {}^{-\theta}y\right) \leq U^{ICAP^*}\left(V, {}^{-\theta}y\right)$ where the first equality comes from the fact that $\left\{c_{i_{j_n}}\right\}_{n=1}^{\infty}$ is a subsequence of $\left\{c_{i_j}\right\}_{j=1}^{\infty}$ and $\lim_{j\to\infty} U_0\left(c_{i_j}, {}^{-\theta}y\right) = \overline{U^{ICAP^*}_{-\theta y}}$, the second follows from the continuity of $U_0\left(., {}^{-\theta}y\right)$ and the third obtains directly from $c \in \Gamma^{ICAP}_{-\theta y}(V)$ and the definition of $U^{ICAP^*}\left(V, {}^{-\theta}y\right)$. Therefore, $U^{ICAP^*}\left(., {}^{-\theta}y\right)$ is usc on $\left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$.

Regarding the boundedness of $U^{ICAP^*}\left(., {}^{-\theta}y\right)$, note that for $\forall V \in \left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$, $U^{ICAP^*}\left(V, {}^{-\theta}y\right) = U_0\left(c_V, {}^{-\theta}y\right)$ for some $c_V \in \Gamma^{ICAP}_{-\theta y}(V) \subset \Omega_{-\theta y}$ with $\Omega_{-\theta y}$ non-empty and compact. Since $U_0\left(., {}^{-\theta}y\right) : \Omega_{-\theta y} \to \mathbb{R}$ is continuous on a compact set, it is also bounded. Consequently, $U^{ICAP^*}\left(., {}^{-\theta}y\right)$ is bounded on $\left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$.■

Lemma 2: *For $\forall {}^{-\theta}y \in Y^{\theta}$, $\forall V \in \left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$, let c^*_V : (1), (2′), (4), (7) hold after ${}^{-\theta}y$, $V_0\left(c^*_V, {}^{-\theta}y\right) = V$ and $U_0\left(c^*_V, {}^{-\theta}y\right) = U^{ICAP^*}\left(V, {}^{-\theta}y\right)$. Then, $\forall {}^{-\theta}y^{\tau-1} \in {}^{-\theta}y \times Y^{\tau}$, $\forall \tau \in \mathbb{Z}_{+}$, $U_\tau\left(c^*_V, {}^{-\theta}y^{\tau-1}\right) = U^{ICAP^*}\left(V_\tau\left(c^*_V, {}^{-\theta}y^{\tau-1}\right), {}^{-\theta}\widetilde{y}\right)$, where ${}^{-\theta}\widetilde{y} = {}^{\tau-\theta}y^{\tau-1}$.*

Proof: Note that for $\forall {}^{-\theta}y^{\tau-1} \in {}^{-\theta}y \times Y^{\tau}$, $\forall \tau \in \mathbb{Z}_{+}$, we have that $V_\tau\left(c^*_V, {}^{-\theta}y^{\tau-1}\right) \in \left\{V^{ICAP}\left({}^{-\theta}\widetilde{y}\right)\right\}$ and therefore, $U^{ICAP^*}\left(V_\tau\left(c^*_V, {}^{-\theta}y^{\tau-1}\right), {}^{-\theta}\widetilde{y}\right)$ is well defined. Since for $\tau = 0$, the result is trivial, fix an arbitrary $\tau \in \mathbb{Z}_{++}$ and ${}^{-\theta}y^{\tau-1} \in {}^{-\theta}y \times Y^{\tau}$ and assume that the lemma does not hold. Then $\exists c'$: (1), (2′), (4), (7) hold after ${}^{-\theta}\widetilde{y}$, $V_0\left(c', {}^{-\theta}\widetilde{y}\right) = V_\tau\left(c^*_V, {}^{-\theta}y^{\tau-1}\right)$ and $U_0\left(c', {}^{-\theta}\widetilde{y}\right) > U_\tau\left(c^*_V, {}^{-\theta}y^{\tau-1}\right)$. Let's construct a supercontract c'' : $\left(a''_t\left({}^{-\theta}y^{t-1}\right), w''_t\left({}^{-\theta}y^t\right)\right) = \left(a'_{t-\tau}\left({}^{-\theta}\widetilde{y}^{t-\tau-1}\right), w'_{t-\tau}\left({}^{-\theta}\widetilde{y}^{t-\tau}\right),\right)$, $\forall {}^{-\theta}y^t = \left({}^{-\theta}y^{t-1}, y_t\right) \in {}^{-\theta}y^{\tau} \times Y^{t-\tau}$: ${}^{\tau}y^t = \widetilde{y}^{t-\tau}$, $\forall t \in \mathbb{Z}_{++}$: $t \geq \tau$, with $\left(a''_t\left(y^{t-1}\right), w''_t\left(y^t\right)\right) = \left(a^*_{V,t}\left(y^{t-1}\right), w^*_{V,t}\left(y^t\right)\right)$ otherwise. By the definition of c^*_V and the construction of c'' we have that c'' satisfies (1), (2′), (4), (7) after ${}^{-\theta}y$ and $V_0\left(c'', {}^{-\theta}y\right) = V_0\left(c^*_V, {}^{-\theta}y\right) = V$. Then, $U_0\left(c'', {}^{-\theta}y\right) \in \left\{U^{ICAP}\left(V, {}^{-\theta}y\right)\right\}$. However, since $U_\tau\left(c'', {}^{-\theta}y^{\tau-1}\right) > U_\tau\left(c^*_V, {}^{-\theta}y^{\tau-1}\right)$, we have that $U_0\left(c'', {}^{-\theta}y\right) > U_0\left(c^*_V, {}^{-\theta}y\right)$, which contradicts the fact that $U_0\left(c^*_V, {}^{-\theta}y\right) = U^{ICAP^*}\left(V, {}^{-\theta}y\right)$.■

The lemma says that at any contingency, the expected discounted utility of the principal who has signed the ICAP supercontract maximizing his/her utility at period 0 while guaranteeing the agent particular initial expected discounted utility also gives the maximum initial utility the principal can get by signing a new ICAP supercontract guaranteeing the agent an initial utility equal to the utility the agent would receive in that contingency under the previous contract. In other words, at the optimum the principal can neither lose nor gain by breaching the original contract and signing a new one guaranteeing the same utility stream to the agent.

For $\forall^{-\theta}y \in Y^{\theta}$ and $\forall V_{-\theta_y} \in \left\{V^{ICAP}\left(^{-\theta}y\right)\right\}$, define $\Gamma_R^{ICAP}\left(V_{-\theta_y},^{-\theta}y\right) := \left\{\left\{a_{-\theta_y}\left(V_{-\theta_y}\right), w_{-\theta_y}\left(V_{-\theta_y}, y\right), V_{+\ -\theta_y}\left(V_{-\theta_y}, y\right)\right\}_{y\in Y} : (8)-(12) \text{ hold}\right\}$.

Proof of Proposition 8: Take an arbitrary $V \in \left\{V^{ICAP}\right\}$. Fix $^{-\theta}y \in Y^{\theta}$. Given the existence of $U^{ICAP^*}\left(V_{-\theta_y},^{-\theta}y\right)$, $\exists\left(a_{-\theta_y}, w_{-\theta_y}\right)$: (1), (2′), (4), (7) hold after $^{-\theta}y$, $V_0\left(a_{-\theta_y}, w_{-\theta_y},^{-\theta}y\right) = V_{-\theta_y}$ and $U_0\left(a_{-\theta_y}, w_{-\theta_y},^{-\theta}y\right) := U^{ICAP^*}\left(V_{-\theta_y},^{-\theta}y\right)$. For $\forall y \in Y$, let $a_{-\theta_y}\left(V_{-\theta_y}\right) := a_0\left(^{-\theta}y\right)$, $w_{-\theta_y}\left(V_{-\theta_y}, y\right) := w_0\left(^{-\theta}y, y\right)$, and $V_{+\ -\theta_y}\left(V_{-\theta_y}, y\right) := V_1\left(a_{-\theta_y}, w_{-\theta_y}, \left(^{-\theta}y, y\right)\right)$. Then, we immediately have that (11) holds. Moreover, (1) $\Rightarrow$ (8), (2′) $\Rightarrow$ (9), (7) $\Rightarrow$ (10). As in the proof of Proposition 4 (a), for $\forall y \in Y$, we can construct $\left(a'_{-\theta+1_{y,y}}, w'_{-\theta+1_{y,y}}\right)$: (1), (2′), (4), (7) hold after $\left(^{-\theta+1}y, y\right)$ and $V_0\left(a'_{-\theta+1_{y,y}}, w'_{-\theta+1_{y,y}}, \left(^{-\theta+1}y, y\right)\right) = V_1\left(a_{-\theta_y}, w_{-\theta_y}, \left(^{-\theta}y, y\right)\right)$, from where we have that (12) is satisfied. Furthermore, by Lemma 2 $\forall y \in Y$, $U_1\left(a_{-\theta_y}, w_{-\theta_y}, \left(^{-\theta}y, y\right)\right) = U^{ICAP^*}\left(V_1\left(a_{-\theta_y}, w_{-\theta_y}, \left(^{-\theta}y, y\right)\right), \left(^{-\theta+1}y, y\right)\right) =$ $= U^{ICAP^*}\left(V_{+\ -\theta_y}\left(V_{-\theta_y}, y\right), \left(^{-\theta+1}y, y\right)\right)$, where $U^{ICAP^*}\left(.,^{-\theta}y\right)$ is usc and bounded from Proposition 7. Then, $U^{ICAP^*}\left(V_{-\theta_y},^{-\theta}y\right) =$ $= U_0\left(a_{-\theta_y}, w_{-\theta_y},^{-\theta}y\right) = \sum\limits_{y\in Y}[\,y - w_0\left(^{-\theta}y, y\right) +$ $+\beta_P U_1\left(a_{-\theta_y}, w_{-\theta_y}, \left(^{-\theta}y, y\right)\right)]\pi\left(y|a_0\left(^{-\theta}y\right)\right) = \sum\limits_{y\in Y}[y - w_{-\theta_y}\left(V_{-\theta_y}, y\right) +$ $+\beta_P U^{ICAP^*}\left(V_{+\ -\theta_y}\left(V_{-\theta_y}, y\right), \left(^{-\theta+1}y, y\right)\right)]\pi\left(y|a_{-\theta_y}(V)\right)$. Then, by the definition of $T(.)$, we have that $T_{-\theta_y}\left(U^{ICAP^*}\right)_{\left(V_{-\theta_y}\right)} \geq U^{ICAP^*}\left(V_{-\theta_y},^{-\theta}y\right)$. Since $^{-\theta}y$ and $V = \left\{V_{-\theta_y}\left(\underline{V}\right)\right\}_{-\theta_y \in Y^{\theta}}$ were chosen randomly, the result generalizes to $T\left(U^{ICAP^*}\right) \geq U^{ICAP^*}$.

Fix an arbitrary $V \in \left\{V^{ICAP}\right\}$ and $^{-\theta}y \in Y^{\theta}$. We have demonstrated above that $\Gamma_R^{ICAP}\left(V_{-\theta_y},^{-\theta}y\right) \neq \emptyset$. Then, since $\Gamma_R^{ICAP}\left(V_{-\theta_y},^{-\theta}y\right)$ can be shown to be compact and U^{ICAP^*} is upper semicontinuous and bounded, $\exists\{a^*_{-\theta_y}\left(V_{-\theta_y}\right), w^*_{-\theta_y}\left(V_{-\theta_y}, y\right), V^*_{+\ -\theta_y}\left(V_{-\theta_y}, y\right)_{y\in Y}$: (8) $-$ (12) hold and $T_{-\theta_y}\left(U^{ICAP^*}\right)_{\left(V_{-\theta_y}\right)} = \sum\limits_{y\in Y}[\,y - w^*_{-\theta_y}\left(V_{-\theta_y}, y\right) +$ $+\beta_P U^{ICAP^*}\left(V^*_{+\ -\theta_y}\left(V_{-\theta_y}, y\right), \left(^{-\theta+1}y, y\right)\right)]\pi\left(y|a^*_{-\theta_y}(V)\right)\}$. By (12), for $\forall y \in Y$, $V^*_{+\ -\theta_y}\left(V_{-\theta_y}, y\right) \in \left\{V^{ICAP}\left(^{-\theta+1}y, y\right)\right\}$, from where $\exists\left(c^*_y\right)$: (1), (2′), (4), (7) hold after $\left(^{-\theta+1}y, y\right)$, $V_0\left(c^*_y, \left(^{-\theta+1}y, y\right)\right) = V^*_{+\ -\theta_y}\left(V_{-\theta_y}, y\right)$ and

$U_0\left(c_y^*, \left({}^{-\theta+1}y, y\right)\right) := U^{ICAP^*}\left(V^*_{+\ {}^{-\theta}y}\left(V_{{}^{-\theta}y}, y\right), \left({}^{-\theta+1}y, y\right)\right)$. Then, define c^{**} : $\left(a_0^{**}\left({}^{-\theta}y\right), w_0^{**}\left({}^{-\theta}y, y\right)\right) = \left(a^*_{{}^{-\theta}y}\left(V_{{}^{-\theta}y}\right), w^*_{{}^{-\theta}y}\left(V_{{}^{-\theta}y}, y\right)\right)$, $\forall y \in Y$, and $\left(a_\tau^{**}\left({}^{-\theta}y^{\tau-1}\right), w_\tau^{**}\left({}^{-\theta}y^{\tau}\right)\right) = \left(a^*_{y,\tau-1}\left({}^{-\theta}\widetilde{y}^{\tau-2}\right), w^*_{y,\tau-1}\left({}^{-\theta}\widetilde{y}^{\tau-1}\right)\right)$, $\forall {}^{-\theta}y^{\tau} = \left({}^{-\theta}y^{\tau-1}, y_\tau\right) \in \left({}^{-\theta}y, y\right) \times Y^\tau : {}^{1}y^{\tau} = \widetilde{y}^{\tau-1}$, $\forall \tau \in \mathbb{Z}_{++}$, $\forall y \in Y$. It is immediate that c^{**} satisfies (1), (2′), (4), (7) after $\left({}^{-\theta}y, y\right)$, $\forall y \in Y$. Moreover, (8) $\Rightarrow a_0^{**}\left({}^{-\theta}y\right) \in A$, (9) $\Rightarrow w_0^{**}\left({}^{-\theta}y, y\right) \in [\underline{w}, \overline{w}]$, $\forall y \in Y$. By construction and (10), we have that (7) holds at ${}^{-\theta}y$. By (11), we obtain that $V_0\left(c^{**}, {}^{-\theta}y\right)$ $= V_{{}^{-\theta}y} \in \left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$, from where (4) is satisfied at ${}^{-\theta}y$. Finally, we have that $U_0\left(c^{**}, {}^{-\theta}y\right) = T\left(U^*_{w^H}\left(V; \underline{V}\right)\right)$. Therefore, $T_{{}^{-\theta}y}\left(U^{ICAP^*}\right)_{\left(V_{{}^{-\theta}y}\right)} \in \left\{U^{ICAP}\left(V_{{}^{-\theta}y}, {}^{-\theta}y\right)\right\}$, from where $U^{ICAP^*}\left(V_{{}^{-\theta}y}, {}^{-\theta}y\right) \geq T_{{}^{-\theta}y}\left(U^{ICAP^*}\right)_{\left(V_{{}^{-\theta}y}\right)}$. As before, this immediately generalizes to $T\left(U^{ICAP^*}\right) \geq U^{ICAP^*}$.■

Proof of Proposition 9: (a) Analogously to the proof of Lemma 1, we can show that for $\forall {}^{-\theta}y \in Y^\theta$, $\Gamma_R^{ICAP}(.)$ is uhc on $\left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$. Then, following an argument similar to the proof of Proposition 7, we conclude that $T(U)_{(.)}$ is usc on $\left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$. It is trivial to show that $T(U)_{(.)}$ is also bounded.

(b) The result follows by the argument of Theorem 3.3 in Stokey and Lucas (1979), p. 54 since it is trivial that T satisfies the Blackwell's sufficient conditions.

(c) Assume on the contrary that $\mu\left(\widetilde{U}, U^{ICAP^*}\right) > 0$. We have that $\mu\left(\widetilde{U}, U^{ICAP^*}\right) = \mu\left(T\left(\widetilde{U}\right), T\left(U^{ICAP^*}\right)\right) \leq \beta_P \mu\left(\widetilde{U}, U^{ICAP^*}\right)$, where the equality follows from the fact that both $\widetilde{U}$ and U^{ICAP^*} are fixed points of T (the first - by assumption, the second - by Proposition 7) and the inequality obtains by (b). However, this contradicts $\beta_P \in (0,1)$. Consequently, $\mu\left(\widetilde{U}, U^{ICAP^*}\right) = 0$.

(d) Since T maps $USCB\left(\left\{V^{ICAP,w^H}\right\}, \mathbb{R}\right)$ into itself, the existence of $T^n(U)$ is guaranteed for $\forall n \in \mathbb{Z}_+$. Using Proposition 8 and successively applying (b), we obtain $\mu\left(T^n(U), U^{ICAP^*}\right) \leq \beta_P^n \mu\left(U, U^{ICAP^*}\right)$. Note that $\mu\left(U, U^{ICAP^*}\right) < \infty$ since U is bounded by assumption and U^{ICAP^*} is bounded by Proposition 7. Therefore, given $\beta_P \in (0,1)$, the result follows.■

Proof of Proposition 10: Just take $U \in \left\{USCBA\left(\left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}, \mathbb{R} \cup \{-\infty\}\right)\right\}_{{}^{-\theta}y \in Y^\theta}$, ${}^{-\theta}y \in Y^\theta$, $V_\infty \in \left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$ and $\{V_i\}_{i=1}^\infty$ s.t. $V_i \in \left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$, $\forall i \in \mathbb{Z}_+$ and $V_i \underset{i\to\infty}{\to} V_\infty$. If $\overline{\lim_i} \underline{T}_{{}^{-\theta}y}(U)_{(V_i)} = -\infty$, the result is trivial. If $\overline{\lim_i} \underline{T}_{{}^{-\theta}y}(U)_{(V_i)} > -\infty$, we can always extract a subsequence $\{V_{i_k}\}_{k=1}^\infty$ of $\{V_i\}_{i=1}^\infty$ s.t. $\underline{T}_{{}^{-\theta}y}(U)_{\left(V_{i_k}\right)} > -\infty$, $\forall k \in \mathbb{Z}_+$ and $\lim_{k\to\infty} \underline{T}_{{}^{-\theta}y}(U)_{\left(V_{i_k}\right)} = \overline{\lim_i} \underline{T}_{{}^{-\theta}y}(U)_{(V_i)}$. Since

$\Lambda_R\left(V_{i_k}, U, {}^{-\theta}y\right) \neq \emptyset$, $\forall k \in \mathbb{Z}_+$, we can apply the argument used in the proof of Proposition 7 to obtain $\overline{\lim_i}\underline{T}_{-\theta y}(U)_{(V_i)} \leq \underline{T}_{-\theta y}(U)_{(V_\infty)}$.■

Proof of Proposition 11: (a) Notice that $U^{ICAP^*} \in \left\{USCB\left(\{V^{ICAP}\left({}^{-\theta}y\right)\}, \mathbb{R}\right)\right\}_{{}^{-\theta}y\in Y^\theta} \subset \left\{USCBA\left(\{V^{ICAP}\left({}^{-\theta}y\right)\}, \mathbb{R}\cup\{-\infty\}\right)\right\}_{{}^{-\theta}y\in Y^\theta}$. Then, directly from the definition of T and $\underline{T}$, we have $\underline{T}\left(U^{ICAP^*}\right) \leq T\left(U^{ICAP^*}\right) = U^{ICAP^*}$, where the equality follows from Proposition 8. Since $\underline{T}_{-\theta y}$ is monotonic for $\forall {}^{-\theta}y \in Y^\theta$, $\{D_i\}_{i\in Z_+}$ is a weakly decreasing sequence of bounded from above usc functions, therefore $\exists D_\infty \in \left\{USCBA\left(\{V^{ICAP}\left({}^{-\theta}y\right)\}, \mathbb{R}\cup\{-\infty\}\right)\right\}_{{}^{-\theta}y\in Y^\theta}$: $D_i\left(V_{-\theta y}, {}^{-\theta}y\right) \underset{i\to\infty}{\to} D_\infty\left(V_{-\theta y}, {}^{-\theta}y\right)$, $\forall V_{-\theta y} \in \left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$, $\forall {}^{-\theta}y \in Y^\theta$.

(b) First we are going to prove $\underline{T}\left(D_\infty\right) \geq D_\infty$. Fix ${}^{-\theta}y \in Y^\theta$ and $V_{-\theta y} \in \left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$. Let us assume that $D_\infty\left(V_{-\theta y}, {}^{-\theta}y\right) > -\infty$ because otherwise the result is trivial. Since $D_\infty\left(V_{-\theta y}, {}^{-\theta}y\right)$ is a limit of a weakly decreasing sequence, we have that $D_i\left(V_{-\theta y}, {}^{-\theta}y\right) > -\infty$, $\forall i \in \mathbb{Z}_+$. Consequently, $D_i\left(V_{-\theta y}, {}^{-\theta}y\right) \geq \underline{U}_{-\theta y}$, $\forall i \in \mathbb{Z}_+$ since $D_i\left(V_{-\theta y}, {}^{-\theta}y\right) < \underline{U}_{-\theta y} \Rightarrow \Lambda_R\left(V_{-\theta y}, D_i, {}^{-\theta}y\right) = \emptyset \Rightarrow D_{i+1}\left(V_{-\theta y}, {}^{-\theta}y\right) = -\infty$. This immediately implies that $D_\infty\left(V_{-\theta y}, {}^{-\theta}y\right) \geq \underline{U}_{-\theta y}$. Moreover, $\Gamma_R\left(V_{-\theta y}, D_{i-1}, {}^{-\theta}y\right) \neq \emptyset$, $\forall i \in \mathbb{Z}_{++}$ since if $\Gamma_R\left(V_{-\theta y}, D_{i-1}, {}^{-\theta}y\right) = \emptyset$, we would have $D_i\left(V_{-\theta y}, {}^{-\theta}y\right) = -\infty$. Then, for $\forall i \in \mathbb{Z}_{++}$, since D_{i-1} is usc and $\Gamma_R\left(V_{-\theta y}, D_{i-1}, {}^{-\theta}y\right)$ is compact (trivial given D_{i-1} is usc), $\exists c_i\left(V_{-\theta y}\right) \in \Gamma_R\left(V_{-\theta y}, D_{i-1}, {}^{-\theta}y\right)$: $D_i\left(V_{-\theta y}, {}^{-\theta}y\right) = \sum_{y\in Y}\left[\; y - c_i\left(V_{-\theta y}, y\right) + \right.$ $\left. +\beta_P D_{i-1}\left(V_{+\,i}\left(V_{-\theta y}, y\right), \left({}^{-\theta+1}y, y\right)\right)\right]\pi\left(y|a_i\left(V_{-\theta y}\right)\right) \geq \underline{U}_{-\theta y}$.[14] Since for $\forall i \in \mathbb{Z}_{++}$, $\Gamma_R\left(V_{-\theta y}, D_{i-1}, {}^{-\theta}y\right) \subset \Gamma_R^{ICAP}\left(V_{-\theta y}, {}^{-\theta}y\right)$ and $\Gamma_R^{ICAP}\left(V_{-\theta y}, {}^{-\theta}y\right)$ is compact, $\exists$ a convergent subsequence of $\left\{c_i\left(V_{-\theta y}\right)\right\}_{i=1}^\infty$, $\left\{c_{i_k}\left(V_{-\theta y}\right)\right\}_{k=1}^\infty$, s.t. $c_\infty\left(V_{-\theta y}\right) := \lim_{k\to\infty} c_{i_k}\left(V_{-\theta y}\right) \in \Gamma_R^{ICAP}\left(V_{-\theta y}, {}^{-\theta}y\right)$. Fix an arbitrary $y \in Y$. Then, we have

$$
\begin{aligned}
D_\infty\left(V_{+\,\infty}\left(V_{-\theta y}, y\right), \left({}^{-\theta+1}y, y\right)\right) &= \\
= \lim_{j\to\infty} D_{i_j-1}\left(V_{+\,\infty}\left(V_{-\theta y}, y\right), \left({}^{-\theta+1}y, y\right)\right) &\geq \\
\geq \lim_{j\to\infty}\overline{\lim_k} D_{i_j-1}\left(V_{+\,i_k}\left(V_{-\theta y}, y\right), \left({}^{-\theta+1}y, y\right)\right) &\geq \\
\geq \lim_{j\to\infty}\overline{\lim_k} D_{i_k-1}\left(V_{+\,i_k}\left(V_{-\theta y}, y\right), \left({}^{-\theta+1}y, y\right)\right) &= \\
= \overline{\lim_k} D_{i_k-1}\left(V_{+\,i_k}\left(V_{-\theta y}, y\right), \left({}^{-\theta+1}y, y\right)\right), &
\end{aligned}
$$

where the first equality follows from $\left\{D_{i_j-1}\right\}_{j=1}^\infty$ being a subsequence of a sequence converging to D_∞ by (a), the first inequality - from the upper semicontinuity of D_{i_j-1}, the second inequality - from the fact that $\{D_i\}_{i=0}^\infty$ is weakly decreasing, hence $D_{i_k-1}\left(V_{+\,i_k}\left(V_{-\theta y}, y\right), \left({}^{-\theta+1}y, y\right)\right) \leq D_{i_j-1}\left(V_{+\,i_k}\left(V_{-\theta y}, y\right), \left({}^{-\theta+1}y, y\right)\right)$, $\forall k \geq j$, and the last equality is trivial. Notice that $D_{i_k-1}\left(V_{+\,i_k}\left(V_{-\theta y}, y\right), \left({}^{-\theta+1}y, y\right)\right) \geq \underline{U}_{-\theta+1 y, y}$, $\forall k \in \mathbb{Z}_{++}$ since by construction

[14] Here, I suppress the dependence of $c_i\left(V_{-\theta y}\right)$ on ${}^{-\theta}y$ in order to simplify the notation.

$c_{i_k}\left(V_{-^\theta y}\right) \in \Gamma_R\left(V_{-^\theta y}, D_{i_k-1}, {}^{-\theta} y\right) \neq \emptyset$. Then, we have that $D_\infty\left(V_{+\infty}\left(V_{-^\theta y}, y\right), \left({}^{-\theta+1}y, y\right)\right) \geq \underline{U}_{-^{\theta+1}y,y}$, from where $c_\infty\left(V_{-^\theta y}\right) \in \Gamma_R\left(V_{-^\theta y}, D_\infty, {}^{-\theta} y\right)$. Finally, $\underline{T}_{-^\theta y}\left(D_\infty\right)_{\left(V_{-^\theta y}\right)} = \max\limits_{\substack{c\left(V_{-^\theta y}\right)\in \\ \Gamma_{R,w^H}\left(V_{-^\theta y}, D_\infty, {}^{-\theta} y\right)}} \sum\limits_{y\in Y}[y- - w\left(V_{-^\theta y}, y\right) + \beta_P D_\infty\left(V_+\left(V_{-^\theta y}, y\right), \left({}^{-\theta+1}y, y\right)\right)] \pi\left(y|a\left(V_{-^\theta y}\right)\right) \geq \geq \sum\limits_{y\in Y}[y - w_\infty\left(V_{-^\theta y}, y\right) + +\beta_P D_\infty\left(V_{+\infty}\left(V_{-^\theta y}, y\right), \left({}^{-\theta+1}y, y\right)\right)]\pi\left(y|a_\infty\left(V_{-^\theta y}\right)\right) \geq \overline{\lim\limits_k} \sum\limits_{y\in Y}[y- -w_{i_k}\left(V_{-^\theta y}, y\right) + \beta_P D_{i_k-1}\left(V_{+i_k}\left(V_{-^\theta y}, y\right), \left({}^{-\theta+1}y, y\right)\right)]\pi\left(y|a_{i_k}\left(V_{-^\theta y}\right)\right) = = \overline{\lim\limits_k} D_{i_k}\left(V_{-^\theta y}, {}^{-\theta} y\right) = D_\infty\left(V_{-^\theta y}, {}^{-\theta} y\right)$, where the first equality follows from the fact that $\Gamma_R\left(V_{-^\theta y}, D_\infty, {}^{-\theta} y\right)$ is non-empty, $D_\infty\left(V_{-^\theta y}, {}^{-\theta} y\right) \geq \underline{U}_{-^\theta y}$, D_∞ is usc and $\Gamma_R\left(V_{-^\theta y}, D_\infty, {}^{-\theta} y\right)$ is compact, the first inequality - from $c_\infty\left(V_{-^\theta y}\right) \in \Gamma_R\left(V_{-^\theta y}, D_\infty, {}^{-\theta} y\right)$, the second inequality - by using the result obtained earlier by developing for $D_\infty\left(V_{+\infty}\left(V_{-^\theta y}, y\right), \left({}^{-\theta+1}y, y\right)\right)$, the following equality - by construction, and the last equality - by construction and (a).

To conclude the proof, we need to show that $\underline{T}\left(D_\infty\right) \leq D_\infty$. Fix ${}^{-\theta}y \in Y^\theta$ and $V_{-^\theta y} \in \left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$. If $\underline{T}_{-^\theta y}\left(D_\infty\right)_{\left(V_{-^\theta y}\right)} = -\infty$, the result is trivial, so assume $\underline{T}_{-^\theta y}\left(D_\infty\right)_{\left(V_{-^\theta y}\right)} > -\infty \Rightarrow \Lambda_R\left(V_{-^\theta y}, D_\infty, {}^{-\theta} y\right) \neq \emptyset$. From (a), we have that for $\forall i \in \mathbb{Z}_+$, $D_\infty \leq D_i$, from where $\Lambda_R\left(V_{-^\theta y}, D_\infty, {}^{-\theta} y\right) \subset \Lambda_R\left(V_{-^\theta y}, D_i, {}^{-\theta} y\right)$, $\forall i \in \mathbb{Z}_+$. Then, for $\forall i \in \mathbb{Z}_+$, $\underline{T}_{-^\theta y}\left(D_\infty\right)_{\left(V_{-^\theta y}\right)} = \max\limits_{\substack{c\left(V_{-^\theta y}\right)\in \\ \Lambda_R\left(V_{-^\theta y}, D_\infty, {}^{-\theta} y\right)}} \sum\limits_{y\in Y}[y - w\left(V_{-^\theta y}, y\right) + +\beta_P D_\infty\left(V_+\left(V_{-^\theta y}, y\right), \left({}^{-\theta+1}y, y\right)\right)]\pi\left(y|a\left(V_{-^\theta y}\right)\right) \leq \max\limits_{\substack{c\left(V_{-^\theta y}\right)\in \\ \Lambda_R\left(V_{-^\theta y}, D_i, {}^{-\theta} y\right)}} \sum\limits_{y\in Y}[y- -w\left(V_{-^\theta y}, y\right) + \beta_P D_i\left(V_+\left(V_{-^\theta y}, y\right), \left({}^{-\theta+1}y, y\right)\right)]\pi\left(y|a\left(V_{-^\theta y}\right)\right) = = D_{i+1}\left(V_{-^\theta y}, {}^{-\theta} y\right)$. Consequently, $\underline{T}_{-^\theta y}\left(D_\infty\right)_{\left(V_{-^\theta y}\right)} \leq \lim\limits_{i\to\infty} D_{i+1}\left(V_{-^\theta y}, {}^{-\theta} y\right) = D_\infty\left(V_{-^\theta y}, {}^{-\theta} y\right)$.

(c) Let $D' \in \left\{USCBA\left(\{V^{ICAP}\left({}^{-\theta}y\right)\}, \mathbb{R}\cup\{-\infty\}\right)\right\}_{-^\theta y\in Y^\theta} : \underline{T}\left(D'\right) = D'$. Note that $\exists \overline{D} \in \left\{USCB\left(\{V^{ICAP}\left({}^{-\theta}y\right)\}, \mathbb{R}\right)\right\}_{-^\theta y\in Y^\theta} : \overline{D} \geq D'$. Consequently, $T\left(\overline{D}\right) \geq T\left(D'\right) \geq D'$, where the first inequality follows from the monotonicity of T, while the second comes from $T \geq \underline{T}$ and the fact that $\underline{T}\left(D'\right) = D'$. Repeating the argument, we obtain $T^n\left(\overline{D}\right) \geq D'$ for $\forall n \in \mathbb{Z}_+$. Then, by Proposition 9 (d) we have that $U^{ICAP^*} = \lim\limits_{n\to\infty} T_U^n\left(\overline{D}\right) \geq D'$, where the convergence is in terms of μ. Fix ${}^{-\theta}y \in Y^\theta$ and $V_{-^\theta y} \in \{V^{ICAP}\left({}^{-\theta}y\right)\}$. By the monotonicity of $\underline{T}$, we have $D_i = \underline{T}^i\left(U^{ICAP^*}\right) \geq \underline{T}^i\left(D'\right) = D'$, $\forall i \in \mathbb{Z}_+$. Therefore, $D_\infty \geq D'$.■

Lemma 3: $\underline{T}\left(\widehat{U}^*\right) \geq \widehat{U}^*$.

Proof: (Adapted from the first part of the proof of Proposition 8) Fix an arbitrary ${}^{-\theta}y \in Y^{\theta}$. If $V_{-\theta_y} \in \{V^{ICAP}\left({}^{-\theta}y\right)\}\backslash\{V^{IC2P}\left({}^{-\theta}y\right)\}$, $\widehat{U}^*\left(V_{-\theta_y},{}^{-\theta}y\right) = -\infty$ and the result is trivial. Therefore, take $V_{-\theta_y} \in \{V^{IC2P}\left({}^{-\theta}y\right)\}$. Then, $\widehat{U}^*\left(V_{-\theta_y},{}^{-\theta}y\right) = U^*\left(V_{-\theta_y},{}^{-\theta}y\right)$. Given the existence of $U^*\left(V_{-\theta_y},{}^{-\theta}y\right)$, we have that $\exists c :$ (1), (2), (4), (5), (7) hold after ${}^{-\theta}y$, $V_0\left(c,{}^{-\theta}y\right) = V_{-\theta_y}$ and $U_0\left(c,{}^{-\theta}y\right) = U^*\left(V_{-\theta_y},{}^{-\theta}y\right)$. By Proposition 3, we have that ($2'$) holds. For $\forall y \in Y$, let $a\left(V_{-\theta_y}\right) := a_0\left({}^{-\theta}y\right)$, $w\left(V_{-\theta_y}, y\right) := w_0\left({}^{-\theta}y, y\right)$, $V_+\left(V_{-\theta_y}, y\right) := V_1\left(c, \left({}^{-\theta}y, y\right)\right)$. Then we immediately have that (11) holds. Moreover, (1) $\Rightarrow$ (8), ($2'$) $\Rightarrow$ (9), (7) $\Rightarrow$ (10). For $\forall y \in Y$, we can construct $c' : (1), (2'), (4), (5), (7)$ hold after $\left({}^{-\theta+1}y, y\right)$ and $V_0\left(c', \left({}^{-\theta+1}y, y\right)\right) = V_1\left(c, \left({}^{-\theta}y, y\right)\right)$, from where we have that $V_+\left(V_{-\theta_y}, y\right) \in \{V^{IC2P}\left({}^{-\theta+1}y, y\right)\} \subset \{V^{ICAP}\left({}^{-\theta+1}y, y\right)\}$, i.e. (12) holds. From (5), we have $U^*\left(V_{-\theta_y},{}^{-\theta}y\right) \geq \underline{U}_{-\theta_y}$. Furthermore, by slightly modifying the argument of Lemma 2, we have that $\forall y \in Y$, $U^*\left(V_+\left(V_{-\theta_y}, y\right), \left({}^{-\theta+1}y, y\right)\right) = U^*\left(V_1\left(c, \left({}^{-\theta}y, y\right)\right), \left({}^{-\theta+1}y, y\right)\right) = U_1\left(c, \left({}^{-\theta}y, y\right)\right)$. Then, (5) $\Rightarrow$ $U^*\left(V_+\left(V_{-\theta_y}, y\right), \left({}^{-\theta+1}y, y\right)\right) \geq \underline{U}_{-\theta+1_{y,y}}$, $\forall y \in Y$. Finally, $\widehat{U}^*$ is usc (by the argument used in the proof of Proposition 7 given the qualifications stated in the proof of Proposition 10) and bounded from above. Then, by the definition of $\underline{T}$, we have that $\underline{T}_{-\theta_y}\left(\widehat{U}^*\right)_{\left(V_{-\theta_y}\right)} \geq \widehat{U}^*\left(V_{-\theta_y},{}^{-\theta}y\right)$.■

Lemma 4: $\underline{T}\left(\widehat{U}^*\right) \leq \widehat{U}^*$.

Proof: (Adapted from the second part of the proof of Proposition 8) Take ${}^{-\theta}y \in Y^{\theta}$. If $V_{-\theta_y} \in \{V^{ICAP}\left({}^{-\theta}y\right)\}\backslash\{V^{IC2P}\left({}^{-\theta}y\right)\}$, $\widehat{U}^*\left(V_{-\theta_y},{}^{-\theta}y\right) = -\infty$ and by the definition of $\underline{T}$ we have $\underline{T}_{-\theta_y}\left(\widehat{U}^*\right)_{\left(V_{-\theta_y}\right)} = -\infty$. What remains is to prove the result on $\{V^{IC2P}\left({}^{-\theta}y\right)\}$. Fix an arbitrary $V_{-\theta_y} \in \left\{V^{IC2P}\left({}^{-\theta}y\right)\right\}$ and note that $\widehat{U}^*\left(V_{-\theta_y},{}^{-\theta}y\right) = U^*\left(V_{-\theta_y},{}^{-\theta}y\right) \geq \underline{U}_{-\theta_y}$. If $\underline{T}_{-\theta_y}\left(\widehat{U}^*\right)_{\left(V_{-\theta_y}\right)} \leq \underline{U}_{-\theta_y}$, the result is trivial. Therefore, assume $\underline{T}_{-\theta_y}\left(\widehat{U}^*\right)_{\left(V_{-\theta_y}\right)} > \underline{U}_{-\theta_y}$. Then, $\underline{T}_{-\theta_y}\left(\widehat{U}^*\right)_{\left(V_{-\theta_y}\right)} = \max\limits_{\substack{c\left(V_{-\theta_y}\right) \in \\ \Gamma_R\left(V_{-\theta_y}, \widehat{U}^*, {}^{-\theta}y\right)}} \sum\limits_{y \in Y}[y - w\left(V_{-\theta_y}, y\right) + \beta_P \widehat{U}^*\left(V_+\left(V_{-\theta_y}, y\right), \left({}^{-\theta+1}y, y\right)\right)]\pi\left(y|a\left(V_{-\theta_y}\right)\right)$ with $\Gamma_R\left(V_{-\theta_y}, \widehat{U}^*,{}^{-\theta}y\right) \neq \emptyset$. Given $\widehat{U}^*$ is usc and $\Gamma_R\left(V_{-\theta_y}, \widehat{U}^*,{}^{-\theta}y\right)$ is compact, $\exists c^*\left(V_{-\theta_y}\right) : (8) - (12)$ hold, $\widehat{U}^*\left(V_+^*\left(V_{-\theta_y}, y\right), \left({}^{-\theta+1}y, y\right)\right) \geq \underline{U}_{-\theta+1_{y,y}}$, $\forall y \in Y$ and $\underline{T}_{-\theta_y}\left(\widehat{U}^*\right)_{\left(V_{-\theta_y}\right)} = \sum\limits_{y \in Y}[y - w^*\left(V_{-\theta_y}, y\right) + \beta_P \widehat{U}^*\left(V_+^*\left(V_{-\theta_y}, y\right), \left({}^{-\theta+1}y, y\right)\right)]\pi\left(y|a^*\left(V_{-\theta_y}\right)\right)$. By (12), $V_+^*\left(V_{-\theta_y}, y\right) \in \left\{V^{ICAP}\left({}^{-\theta+1}y, y\right)\right\}$, which together with $\widehat{U}^*\left(V_+^*\left(V_{-\theta_y}, y\right), \left({}^{-\theta+1}y, y\right)\right) \geq \underline{U}_{-\theta+1_{y,y}}$ implies that $V_+^*\left(V_{-\theta_y}, y\right) \in \left\{V^{IC2P}\left({}^{-\theta+1}y, y\right)\right\}$. Since $\widehat{U}^*\left(V_+^*\left(V_{-\theta_y}, y\right), \left({}^{-\theta+1}y, y\right)\right) = U^*\left(V_+^*\left(V_{-\theta_y}, y\right), \left({}^{-\theta+1}y, y\right)\right)$, $\exists c_y^* :$ (1), (2), (4), (5), (7) hold after $\left({}^{-\theta+1}y,y\right)$, $V_0\left(c_y^*,\left({}^{-\theta+1}y,y\right)\right) = V_+^*\left(V_{-\theta_y},y\right)$ and $U_0\left(c_y^*,\left({}^{-\theta+1}y,y\right)\right) := \widehat{U}^*\left(V_+^*\left(V_{-\theta_y},y\right),\left({}^{-\theta+1}y,y\right)\right)$. Note that this is true for $\forall y \in Y$. Then, define c^{**} : $\left(a_0^{**}\left({}^{-\theta}y\right),w_0^{**}\left({}^{-\theta}y,y\right)\right) = \left(a^*\left(V_{-\theta_y}\right),w^*\left(V_{-\theta_y},y\right)\right)$, $\forall y \in Y$, and $\left(a_\tau^{**}\left({}^{-\theta}y^{\tau-1}\right),w_\tau^{**}\left({}^{-\theta}y^\tau\right)\right) = \left(a_{y,\tau-1}^*\left({}^{-\theta}\widetilde{y}^{\tau-2}\right),w_{y,\tau-1}^*\left({}^{-\theta}\widetilde{y}^{\tau-2},\widetilde{y}_{\tau-1}\right)\right)$, $\forall{}^{-\theta}y^\tau = \left({}^{-\theta}y^{\tau-1},y_\tau\right) \in \left({}^{-\theta}y,y\right)\times Y^\tau : {}^1y^\tau = \widetilde{y}^\tau$, $\forall\tau\in\mathbb{Z}_{++}$, $\forall y\in Y$.

It is immediate that c^{**} satisfies (1), (2), (4), (5), (7) after $\left({}^{-\theta}y,y\right)$, $\forall y\in Y$. Moreover, (8) $\Rightarrow a_0^{**}\left({}^{-\theta}y\right)\in A$, (9) $\Rightarrow w_0^{**}\left({}^{-\theta}y,y\right)\geq \underline{w}$, $\forall y\in Y$. By construction and (10), we have that (7) holds at ${}^{-\theta}y$. By (11), we obtain that $V_0\left(c^{**},{}^{-\theta}y\right) = V_{-\theta_y} \in \left\{V^{IC2P}\left({}^{-\theta}y\right)\right\}$, from where (4) is satisfied at ${}^{-\theta}y$. Furthermore, we have that $U_0\left(c^{**},{}^{-\theta}y\right) = T_{-\theta_y}\left(\widehat{U}^*\right)_{\left(V_{-\theta_y}\right)} > \underline{U}_{-\theta_y}$. Therefore, $T_{-\theta_y}\left(\widehat{U}^*\right)_{\left(V_{-\theta_y}\right)} \in \left\{U^{IC2P}\left(V_{-\theta_y},{}^{-\theta}y\right)\right\}$, from where $\widehat{U}^*\left(V_{-\theta_y},{}^{-\theta}y\right) = U^*\left(V_{-\theta_y},{}^{-\theta}y\right) \geq T_{-\theta_y}\left(\widehat{U}^*\right)_{\left(V_{-\theta_y}\right)}$.■

Proof of Proposition 12: From Lemmas 3 and 4.■

Proof of Proposition 13: Since $\widehat{U}^* \in \left\{USCBA\left(\{V^{ICAP}\left({}^{-\theta}y\right)\},\mathbb{R}\cup\{-\infty\}\right)\right\}_{-\theta_{y\in Y^\theta}}$, by Propositions 11 (c) and 12 we obtain $\widehat{U}^* \leq D_\infty$. What remains to be shown is that $\widehat{U}^* \geq D_\infty$. Fix ${}^{-\theta}y \in Y^\theta$ and $V_{-\theta_y} \in \{V^{ICAP}\left({}^{-\theta}y\right)\}$. If $D_\infty\left(V_{-\theta_y},{}^{-\theta}y\right) = -\infty$, the result is trivial; therefore, assume $D_\infty\left(V_{-\theta_y},{}^{-\theta}y\right) > -\infty$. Then, $D_\infty\left(V_{-\theta_y},{}^{-\theta}y\right) =$
$= \underline{T}_{-\theta_y}\left(D_\infty\right)_{\left(V_{-\theta_y}\right)} = \max\limits_{\substack{c_{-\theta_y}\left(V_{-\theta_y}\right)\in \\ \Gamma_R\left(V_{-\theta_y},D_\infty,{}^{-\theta}y\right)}} \sum\limits_{y\in Y}[y - w\left(V_{-\theta_y},y\right) +$
$+\beta_P D_\infty\left(V_+\left(V_{-\theta_y},y\right),\left({}^{-\theta+1}y,y\right)\right)]\pi\left(y|a\left(V_{-\theta_y}\right)\right)$ with $\Gamma_R\left(V_{-\theta_y},D_\infty,{}^{-\theta}y\right)$ nonempty and $D_\infty\left(V_{-\theta_y},{}^{-\theta}y\right) \geq \underline{U}_{-\theta_y}$ since otherwise we would have $D_\infty\left(V_{-\theta_y},{}^{-\theta}y\right) = -\infty$. Since D_∞ is usc and $\Gamma_R\left(V_{-\theta_y},D_\infty,{}^{-\theta}y\right)$ is compact, we have that $\exists c_{-\theta_y}^*\left(V_{-\theta_y}\right)\in\Gamma_R\left(V_{-\theta_y},D_\infty,{}^{-\theta}y\right)$ s.t. $\underline{T}_{-\theta_y}\left(D_\infty\right)_{\left(V_{-\theta_y}\right)} =$
$= \sum\limits_{y\in Y}\left[y - w^*\left(V_{-\theta_y},y\right) + \beta_P D_\infty\left(V_+^*\left(V_{-\theta_y},y\right),\left({}^{-\theta+1}y,y\right)\right)\right]\pi\left(y|a^*\left(V_{-\theta_y}\right)\right)$.
Note that from $D_\infty\left(V_+^*\left(V_{-\theta_y},y\right),\left({}^{-\theta+1}y,y\right)\right) \geq \underline{U}_{-\theta+1_{y,y}}$, $\forall y \in Y$, we have $D_\infty\left(V_+^*\left(V_{-\theta_y},y\right),\left({}^{-\theta+1}y,y\right)\right) = \underline{T}_{-\theta+1_{y,y}}\left(D_\infty\right)_{V_+^*\left(V_{-\theta_y},y\right)} =$
$= \max\limits_{\substack{c_{-\theta_{y,y}}\left(V_+^*\left(V_{-\theta_y},y\right)\right)\in \\ \Gamma_R\left(V_+^*\left(V_{-\theta_y},y\right),D_\infty,\left({}^{-\theta+1}y,y\right)\right)}} \sum\limits_{y'\in Y}[y' - w\left(V_+^*\left(V_{-\theta_y},y\right),y'\right) +$
$+ \beta_P D_\infty\left(V_+\left(V_+^*\left(V_{-\theta_y},y\right),y'\right),\left(y^{-\theta+2},y,y'\right)\right)]\pi\left(y'|a\left(V_+^*\left(V_{-\theta_y},y\right)\right)\right)$ with $\Gamma_R\left(V_+^*\left(V_{-\theta_y},y\right),D_\infty,\left({}^{-\theta+1}y,y\right)\right)$ nonempty, so the previous analysis applies. Proceeding in this way, we can construct a supercontract c s.t. $a_\tau\left({}^{-\theta}y^{\tau-1}\right) := a_{-\theta_{y^{\tau-1}}}^*\left(V_+^{*\ \tau}\left(V_{-\theta_y},{}^{-\theta}y^{\tau-1}\right)\right)$, $w_\tau\left({}^{-\theta}y^\tau\right) := w_{-\theta_{y^{\tau-1}}}^*\left(V_+^{*\ \tau}\left(V_{-\theta_y},{}^{-\theta}y^{\tau-1}\right),y_\tau\right)$, $\forall{}^{-\theta}y^\tau = \left({}^{-\theta}y^{\tau-1},y_\tau\right)\in{}^{-\theta}y\times Y^\tau\times Y$, $\forall\tau\in\mathbb{Z}_+$, where $V_+^{*\ \tau}\left(V_{-\theta_y},{}^{-\theta}y^{\tau-1}\right) := V_{+\ \langle y_{\tau-1}\rangle}^*\circ...\circ V_{+\ \langle y_0\rangle}^*\left(V_{-\theta_y},{}^{-\theta}y\right)$, $\tau\in\mathbb{Z}_+$ and

$V_{+}^{*\,0}\left(V_{-\theta_y},{}^{-\theta}y\right) := V_{-\theta_y}$ with $V_{+\ \langle\widetilde{y}\rangle}^{*}\left(\widetilde{V},{}^{-\theta}y^{t-1}\right) := V_{+\ {}^{-\theta}y^{t-1}}^{*}\left(\widetilde{V},\widetilde{y}\right)$, $\forall\widetilde{y}\in Y$, $\forall\widetilde{V} \in \left\{V^{ICAP}\left({}^{t-\theta}y^{t-1}\right)\right\}$, $\forall{}^{-\theta}y^{t-1} \in {}^{-\theta}y \times Y^{t}$, $\forall t \in \mathbb{Z}_{+}$. We immediately have (8) $\Rightarrow$ (1) and (9) $\Rightarrow$ (2). By (11), $V_{\tau}\left(c,{}^{-\theta}y^{\tau-1}\right)=V_{+}^{*\,\tau}\left(V_{-\theta_y},{}^{-\theta}y^{\tau-1}\right)$, $\forall{}^{-\theta}y^{\tau-1} \in {}^{-\theta}y \times Y^{\tau}$, $\forall\tau \in \mathbb{Z}_{+}$ since by (12) $V_{+}^{*\,\tau}\left(V_{-\theta_y},{}^{-\theta}y^{\tau-1}\right) \in \left\{V^{ICAP}\left({}^{\tau-\theta}y^{\tau-1}\right)\right\}$ and is therefore bounded, while $V_{\tau}\left(c,{}^{-\theta}y^{\tau-1}\right)$ is bounded given (8) and (9). In particular, $V_{0}\left(c,{}^{-\theta}y^{\tau-1}\right)=V_{-\theta_y}$. Then (10) $\Rightarrow$ (7). Furthermore, $V_{\tau}\left(c,{}^{-\theta}y^{\tau-1}\right) \in \left\{V^{ICAP}\left({}^{\tau-\theta}y^{\tau-1}\right)\right\}$, $\forall{}^{-\theta}y^{\tau-1} \in {}^{-\theta}y\times Y^{\tau}$, $\forall\tau\in\mathbb{Z}_{+}$ implies that (4) holds. We also have that $U_{\tau}\left(c,{}^{-\theta}y^{\tau-1}\right) = D_{\infty}\left(V_{+}^{*\,\tau}\left(V_{-\theta_y},{}^{-\theta}y^{\tau-1}\right),{}^{\tau-\theta}y^{\tau-1}\right)$, $\forall{}^{-\theta}y^{\tau-1} \in {}^{-\theta}y \times Y^{\tau}$, $\forall\tau \in \mathbb{Z}_{+}$ since $U_{\tau}\left(c,{}^{-\theta}y^{\tau-1}\right)$ is bounded given (8) and (9) and $D_{\infty}\left(.,{}^{\tau-\theta}y^{\tau-1}\right)$ is bounded from above by $\underset{{}^{\tau-\theta}y^{\tau-1}\in Y^{\theta}}{max}\left\{\underset{\widetilde{V}\in\left\{V^{ICAP}\left({}^{\tau-\theta}y^{\tau-1}\right)\right\}}{max}\left\{U^{ICAP^{*}}\left(\widetilde{V},{}^{\tau-\theta}y^{\tau-1}\right)\right\}\right\}$ (well defined by Proposition 7 and Y finite) and from below by $\underline{U}$. In particular, $U_{0}\left(c,{}^{-\theta}y\right) = D_{\infty}\left(V_{-\theta_y},{}^{-\theta}y\right)$. Then, (5) is satisfied at any node. Therefore, $V_{-\theta_y} \in \left\{V^{IC2P}\left({}^{-\theta}y\right)\right\}$ and $D_{\infty}\left(V_{-\theta_y},{}^{-\theta}y\right) \in \left\{U\left(V_{-\theta_y},{}^{-\theta}y\right)\right\}$. Then, $\widehat{U}^{*}\left(V_{-\theta_y},{}^{-\theta}y\right) = U^{*}\left(V_{-\theta_y},{}^{-\theta}y\right) \geq D_{\infty}\left(V_{-\theta_y},{}^{-\theta}y\right)$.■

Lemma 5: $\left\{V^{ICAP}\right\} \subset B\left(\left\{V^{ICAP}\right\}\right)$.

Proof: Let $V \in \left\{V^{ICAP}\right\}$. Fix an arbitrary ${}^{-\theta}y \in Y^{\theta}$. By $V_{-\theta_y} \in \left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$, $\exists c$: (1), (2′), (4), (7) hold after ${}^{-\theta}y$ and $V_{0}\left(w,a,{}^{-\theta}y\right) = V_{-\theta_y}$. By construction $V_{-\theta_y} \in \left[\underline{V}_{-\theta_y},\widehat{V}\right]$. For $\forall y \in Y$, let $a_{-\theta_y}\left(V_{-\theta_y}\right) := a_{0}\left({}^{-\theta}y\right)$, $w_{-\theta_y}\left(V_{-\theta_y},y\right) = w_{0}\left({}^{-\theta}y,y\right)$, and $V_{+\ {}^{-\theta}y}\left(V_{-\theta_y},y\right) := V_{1}\left(c,\left({}^{-\theta}y,y\right)\right)$. Given these choices, we immediately have that (11) holds. Moreover, (1) $\Rightarrow$ (8), (2′) $\Rightarrow$ (9), (7) $\Rightarrow$ (10). Note that for $\forall y \in Y$, $\left\{V^{ICAP}\left({}^{-\theta+1}y,y\right)\right\}\cap\left[\underline{V}_{{}^{-\theta+1}y,y},+\infty\right) = \left\{V^{ICAP}\left({}^{-\theta+1}y,y\right)\right\}$. Since for $\forall y \in Y$ we can construct a supercontract c'_{y} : (1), (2), (4), (7) hold after $\left({}^{-\theta+1}y,y\right)$ and $V_{0}\left(c'_{y},\left({}^{-\theta+1}y,y\right)\right) = V_{1}\left(c,\left({}^{-\theta}y,y\right)\right)$, we have that (12′) is satisfied. Therefore, $V_{-\theta_y} \in B\left(\left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}\right)$. Since ${}^{-\theta}y \in Y^{\theta}$ was chosen randomly, this generalizes to $V \in B\left(\left\{V^{ICAP}\right\}\right)$.■

The lemma establishes that $\left\{V^{ICAP}\right\}$ is self-generating in the terminology of Abreu, Pearce and Stacchetti (1990).

Lemma 6: *Assume* $W = \left\{W_{-\theta_y}\right\}_{{}^{-\theta}y\in Y^{\theta}} : \emptyset \neq W_{-\theta_y} \subset B_{-\theta_y}\left(W\right)$, $\forall{}^{-\theta}y \in Y^{\theta}$*. Then,* $B\left(W\right) \subset \left\{V^{ICAP}\right\}$.

Proof: Let the condition of the Lemma hold and take $V \in B\left(W\right)$. Fix an arbitrary ${}^{-\theta}y \in Y^{\theta}$. Since $V_{-\theta_y} \in B_{-\theta_y}\left(W\right)$, $\exists c_{-\theta_y}\left(V_{-\theta_y}\right)$: (8)-(11) and (12′) hold. By (12′) and $W_{{}^{-\theta+1}y,y} \subset B_{{}^{-\theta+1}y,y}\left(W\right)$, we obtain that $V_{+\ {}^{-\theta}y}\left(V_{-\theta_y},y\right) \in B_{{}^{-\theta+1}y,y}\left(W\right)$. Then, $\forall y \in Y$, $\exists c_{\left({}^{-\theta}y,y\right)}\left(V_{+\ {}^{-\theta}y}\left(V_{-\theta_y},y\right)\right)$:(8)-(11) and (12′) hold. Proceeding this way, we can construct a supercontract $c : a_{\tau}\left({}^{-\theta}y^{\tau-1}\right) :=$

$a_{-\theta y^{\tau-1}}\left(V_+^{\tau}\left(V_{-\theta y},{}^{-\theta}y^{\tau-1}\right)\right)$, $w_\tau\left({}^{-\theta}y^{\tau}\right) := w_{-\theta y^{\tau-1}}\left(V_+^{\tau}\left(V_{-\theta y},{}^{-\theta}y^{\tau-1}\right),y_\tau\right)$, $\forall^{-\theta}y^{\tau}=\left({}^{-\theta}y^{\tau-1},y_\tau\right)\in{}^{-\theta}y\times Y^{\tau}\times Y$, $\forall\tau\in\mathbb{Z}_+$, where $V_+^{\tau}\left(V_{-\theta y},{}^{-\theta}y^{\tau-1}\right):=V_{+\langle y_{\tau-1}\rangle}\circ\ldots\circ V_{+\langle y_0\rangle}\left(V_{-\theta y},{}^{-\theta}y\right)$, $\tau\in\mathbb{Z}_+$ and $V_+^{0}\left(V_{-\theta y},{}^{-\theta}y\right):=V_{-\theta y}$ with $V_{+\langle\widetilde{y}\rangle}\left(\widetilde{V},{}^{-\theta}y^{t-1}\right):=V_{+\ {}^{-\theta}y^{t-1}}\left(\widetilde{V},\widetilde{y}\right)$, $\forall\widetilde{y}\in Y$, $\forall\widetilde{V}\in\left\{V^{ICAP}\left({}^{t-\theta}y^{t-1}\right)\right\}$, $\forall^{-\theta}y^{t-1}\in{}^{-\theta}y\times Y^{t}$, $\forall t\in\mathbb{Z}_+$. By construction, (8)-(11), (12′) hold at $\forall^{-\theta}y^{t-1}\in{}^{-\theta}y\times Y^{\tau}$, $\forall\tau\in\mathbb{Z}_+$ and $V_\tau\left(c,{}^{-\theta}y^{t-1}\right)-$ $-V_+^{\tau}\left(V_{-\theta y},{}^{-\theta}y^{\tau-1}\right)=\lim\limits_{T\to\infty}\beta_A{}^{T}\sum\limits_{y_{\tau+T-1}\in Y}\ldots\sum\limits_{y_\tau\in Y}[V_{\tau+T}\left(w,a,y^{\tau+T-1}\right)-$ $-V_+^{\tau+T}\left(V_{-\theta y},{}^{-\theta}y^{\tau+T-1}\right)]\prod\limits_{i=\tau}^{\tau+T-1}\pi\left(y_i|a_i\left(y^{\tau-1}\right)\right)=0$. In particular, $V_0\left(c,{}^{-\theta}y\right)=V_{-\theta y}$. At every node, we have (8)⇒(1), (9)⇒(2′), (10)⇒(7). For every node, but ${}^{-\theta}y$, (12′) implies (4). Since $V_{-\theta y}\in B_{-\theta y}(W)\subset\left[\underline{V}_{-\theta y},\widehat{V}\right]$, (4) also holds at ${}^{-\theta}y$. Therefore, $V_{-\theta y}\in\left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$, which generalizes to $V\in\left\{V^{ICAP}\right\}$.■

The lemma says that the image of every nonempty, self-generating set is a subset of $\left\{V^{ICAP}\right\}$.

Proof of Proposition 14: (a) By Assumption 3 and Lemma 5, we have that the condition of Lemma 6 holds. Therefore, we obtain $B\left(\left\{V^{ICAP}\right\}\right)\subset\left\{V^{ICAP}\right\}$, which together with Lemma 5 implies the result.

(b) It follows by Lemma 6.■

Lemma 7: *Assume* $W'=\left\{W'_{-\theta y}\right\}_{-\theta y\in Y^{\theta}}$ *and* $W''=\left\{W''_{-\theta y}\right\}_{-\theta y\in Y^{\theta}}$ $: W'_{-\theta y}\subset W'_{-\theta y}\subset\mathbb{R}$, $\forall^{-\theta}y\in Y^{\theta}$. *Then,* $B_{-\theta y}(W')\neq\emptyset\Rightarrow B_{-\theta y}(W')\subset B_{-\theta y}(W'')$, $\forall^{-\theta}y\in Y^{\theta}$.

Proof: Trivial.■

Lemma 8: *Assume* $W=\left\{W_{-\theta y}\right\}_{-\theta y\in Y^{\theta}}$*:* $W_{-\theta y}\subset\mathbb{R}$ *compact,* $\forall^{-\theta}y\in Y^{\theta}$. *Then,* $B_{-\theta y}(W)\neq\emptyset\Rightarrow B_{-\theta y}(W)$ *compact,* $\forall^{-\theta}y\in Y^{\theta}$.

Proof: Let the condition of the Lemma hold and assume $B_{-\theta y}(W)\neq\emptyset$ for some ${}^{-\theta}y\in Y^{\theta}$. Note that $B_{-\theta y}(W)\subset\left[\underline{V}_{-\theta y},\widehat{V}\right]\subset\mathbb{R}$ is bounded by definition. We should also show that it is closed. Take an arbitrary convergent sequence $\{V_i\}_{i\in\mathbb{Z}_{++}}: V_i\in B_{-\theta y}(W)$, $\forall i\in\mathbb{Z}_{++}$ with $V_i\underset{i\to\infty}{\to}V_\infty$. We need to prove that $V_\infty\in B_{-\theta y}(W)$. By construction, we have that for $\forall i\in\mathbb{Z}_{++}$, $V_i\in\left[\underline{V}_{-\theta y},\widehat{V}\right]$ and $\exists c_i$: (8) − (11), (12′) hold for V_i. By $V_i\in\left[\underline{V}_{-\theta y},\widehat{V}\right]$, $\forall i\in\mathbb{Z}_{++}$, we obtain $V_\infty\in\left[\underline{V}_{-\theta y},\widehat{V}\right]$. By (8),(9),(12′) and $\left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}\subset\left[\underline{V}_{-\theta y},\widehat{V}\right]\subset\left[\underline{V},\widehat{V}\right]\subset\mathbb{R}$, $\forall^{-\theta}y\in Y^{\theta}$, we have that $\{c_i\}_{i\in\mathbb{Z}_{++}}$ is uniformly bounded, therefore

$\exists$ a subsequence $\{c_{i_k}\}_{k\in\mathbb{Z}_{++}}$ of $\{c_i\}_{i\in\mathbb{Z}_{++}}$: $c_{i_k} \underset{k\to\infty}{\to} c_\infty$. We immediately have that c_∞ satisfies (8)-(11) for V_∞. Finally, since (12′) is satisfied for $\forall i \in \mathbb{Z}_{++}$ and $\left\{V^{ICAP}\left({}^{-\theta}y\right)\right\}$ is compact by Proposition 4 (a) for $\forall {}^{-\theta}y \in Y^\theta$, we also have that (12′) holds for c_∞.■

Proof of Proposition 15: For $\forall {}^{-\theta}y \in Y^\theta$ and $\forall i \in \mathbb{Z}_+$, denote by $W_i\left({}^{-\theta}y\right)$ the element of W_i corresponding to initial history ${}^{-\theta}y$. By the condition of the Proposition and Assumption 3, we have that $\emptyset \neq \left\{V^{ICAP}\left({}^{-\theta}y\right)\right\} \subset W_0\left({}^{-\theta}y\right) \subset \mathbb{R}$, $\forall {}^{-\theta}y \in Y^\theta$. Since by Proposition 14 (a) and Assumption 3, $B_{{}^{-\theta}y}\left(\left\{V^{ICAP}\right\}\right) = \left\{V^{ICAP}\left({}^{-\theta}y\right)\right\} \neq \emptyset$, $\forall {}^{-\theta}y \in Y^\theta$, we can apply Lemma 7 to obtain $\emptyset \neq \left\{V^{ICAP}\left({}^{-\theta}y\right)\right\} \subset W_1\left({}^{-\theta}y\right) \subset \mathbb{R}$, $\forall {}^{-\theta}y \in Y^\theta$. Using $W_1 \subset W_0$ and repeating the argument, we reach $\left\{V^{ICAP}\right\} \subset W_{i+1} \subset W_i$, $\forall i \in \mathbb{Z}_+$ Then, $\{W_i\}_{i\in\mathbb{Z}_+}$ is a sequence of non-empty, compact (by Lemma 8 since W_0 compact), monotonically decreasing (nested) sets; therefore it converges to $W_\infty = \underset{i\in\mathbb{Z}_+}{\cap} W_i \supset \left\{V^{ICAP}\right\}$ with W_∞ compact. What remains to be shown is that $W_\infty \subset \left\{V^{ICAP}\right\}$. By Lemma 6, it is enough to show that $W_\infty \subset B\left(W_\infty\right)$. Let $V \in W_\infty$. This implies that $V \in W_i$, $\forall i \in \mathbb{Z}_+$. Fix an arbitrary ${}^{-\theta}y \in Y^\theta$. We have that $\exists c_i\left(V_{{}^{-\theta}y}\right)$:(8)-(11), (12′) hold for $V_{{}^{-\theta}y}$. By (8), (9), (12′) and $W_i\left({}^{-\theta}y\right) \subset W_0\left({}^{-\theta}y\right) \subset \mathbb{R}$, $\forall i \in \mathbb{Z}_+$, we have that $\left\{c_i\left(V_{{}^{-\theta}y}\right)\right\}_{i\in\mathbb{Z}_+}$ is uniformly bounded; therefore, $\exists$ a subsequence $\left\{c_{i_k}\left(V_{{}^{-\theta}y}\right)\right\}_{k\in\mathbb{Z}_+}$of $\left\{c_i\left(V_{{}^{-\theta}y}\right)\right\}_{i\in\mathbb{Z}_+}^\infty$: $c_{i_k}\left(V_{{}^{-\theta}y}\right) \underset{k\to\infty}{\to} c_\infty\left(V_{{}^{-\theta}y}\right)$. It is immediate that $c_\infty\left(V_{{}^{-\theta}y}\right)$ satisfies (8)-(11) for $V_{{}^{-\theta}y}$. Moreover, $V_+\left(V_{{}^{-\theta}y}, y\right) \geq \underline{V}_{{}^{-\theta+1}y,y}$, $\forall y \in Y$. We also need to show that for $\forall y \in Y$, $V_{+\,\infty}\left(V_{{}^{-\theta}y}, y\right) \in W_\infty\left({}^{-\theta+1}y, y\right)$. Fix an arbitrary $y \in Y$ and assume, on the contrary, that $V_{+\,\infty}\left(V_{{}^{-\theta}y}, y\right) \notin W_\infty\left({}^{-\theta+1}y, y\right)$. Since $W_\infty\left({}^{-\theta+1}y, y\right) = \underset{i\in\mathbb{Z}_+}{\cap} W_i\left({}^{-\theta+1}y, y\right) = \underset{k\in\mathbb{Z}_+}{\cap} W_{i_k}\left({}^{-\theta+1}y, y\right)$, we have that $\exists k' \in \mathbb{Z}_+ : V_{+\,\infty}\left(V_{{}^{-\theta}y}, y\right) \notin W_{i_{k'}}\left({}^{-\theta+1}y, y\right)$. Furthermore, $\left\{W_{i_{k'}}\right\}_{k\in\mathbb{Z}_+}$ was shown to be a monotonically decreasing (nested) sequence, from where $V_{+\,i_k}\left(V_{{}^{-\theta}y}, y\right) \in W_{i_k} \subset W_{i_{k'}}\left({}^{-\theta+1}y, y\right)$, $\forall k \in \mathbb{Z}_+ : k \geq k'$. Since $W_{i_{k'}}\left({}^{-\theta+1}y, y\right)$ is closed and $V_{+\,i_k}\left(V_{{}^{-\theta}y}, y\right) \underset{k\to\infty}{\to} V_{+\,\infty}\left(V_{{}^{-\theta}y}, y\right)$, we obtain that $V_{+\,\infty}\left(V_{{}^{-\theta}y}, y\right) \in W_{i_{k'}}\left({}^{-\theta+1}y, y\right)$, i.e. a contradiction is reached. This proves $V_{+\,\infty}\left(V_{{}^{-\theta}y}, y\right) \in W_\infty\left({}^{-\theta+1}y, y\right)$, $\forall y \in Y$. Consequently, (12″) holds for $c_\infty\left(V_{{}^{-\theta}y}\right)$. Finally, note that $V_{{}^{-\theta}y} \in \left[\underline{V}_{{}^{-\theta}y}, \widehat{V}\right]$ follows immediately from $V_{{}^{-\theta}y} \in W_1\left({}^{-\theta}y\right)$. Therefore, $V_{{}^{-\theta}y} \in B_{{}^{-\theta}y}\left(W_\infty\right)$, which generalizes to $V \in B\left(W_\infty\right)$.■

For $\forall W = \left\{W_{{}^{-\theta}y}\right\}_{{}^{-\theta}y\in Y^\theta}$: $W_{{}^{-\theta}y} \in \mathbb{R}$, $\forall {}^{-\theta}y \in Y^\theta$ let $B'(W) := \left\{B'_{{}^{-\theta}y}(W)\right\}_{{}^{-\theta}y\in Y^\theta}$ with $B'_{{}^{-\theta}y}(W) := \{V_{{}^{-\theta}y} \in \left[\underline{V}_{{}^{-\theta}y}, \widehat{V}\right] : \exists c_{{}^{-\theta}y}\left(V_{{}^{-\theta}y}\right)$: (8)-(11) and (12″) hold$\}$, where (12″) is defined as $V_{+\,{}^{-\theta}y}\left(V_{{}^{-\theta}y}, y\right) \in W_{{}^{-\theta+1}y,y}$.

Lemma 9: *Take* $W'_0 := \left\{W'_0\left({}^{-\theta}y\right)\right\}_{{}^{-\theta}y\in Y^\theta}$ *with* $W'_0\left({}^{-\theta}y\right) := \left[\underline{V}_{{}^{-\theta}y}, \widehat{V}\right]$, $\forall {}^{-\theta}y \in Y^\theta$ *and let* $W'_{i+1} := B'\left(W'_i\right)$ *for* $\forall i \in \mathbb{Z}_+$. *Then,* $W'_{i+1} \subset W'_i$, $\forall i \in \mathbb{Z}_+$

and $W'_\infty := \lim_{i\to\infty} W'_i = \left\{V^{ICAP}\right\}$.

Proof: We have that W'_0 is compact and $\left\{V^{ICAP}\right\} \subset W'_0 \subset \mathbb{R}^{N^\theta}$. Note that for $\forall W \subset \mathbb{R}^{N^\theta}$: $B_{{}^{-\theta}y}(W) \neq \emptyset$, $B_{{}^{-\theta}y}(W) \subset B'_{{}^{-\theta}y}(W)$. Then, by Lemma 7 and Proposition 14 (a), we obtain $\left\{V^{ICAP}\right\} \subset B(W'_0) \subset B'(W'_0)$. Using the same arguments plus the monotonicity of B' (trivial), we have $\left\{V^{ICAP}\right\} \subset W'_i$, $\forall i \in \mathbb{Z}_+$. Moreover, by construction $B'(W'_0) \subset W'_0$. Then, the condition $B(W'_0) \subset W'_0$ is satisfied. Observe that for $\forall {}^{-\theta}y \in Y^\theta$, $W'_1\left({}^{-\theta}y\right) = \{V_{{}^{-\theta}y} \in \left[\underline{V}_{{}^{-\theta}y}, \widehat{V}\right] : \exists c\left(V_{{}^{-\theta}y}\right)$:(8)-(11), $(12'')\} = \{V_{{}^{-\theta}y} \in \left[\underline{V}_{{}^{-\theta}y}, \widehat{V}\right] : \exists c\left(V_{{}^{-\theta}y}\right)$:(8)-(11), $(12')\} = B_{{}^{-\theta}y}(W'_0)$ since, by construction, $W'_0\left({}^{-\theta+1}y, y\right) \cap \left[\underline{V}_{{}^{-\theta+1}y,y}, +\infty\right) = W'_0\left({}^{-\theta+1}y, y\right)$, $\forall y \in Y$. Furthermore, by $W'_1 \subset W'_0$ and the monotonicity of B', we obtain $W'_{i+1} \subset W'_i$, $\forall i \in \mathbb{Z}_+$. Then, it is trivial that $W'_{i+1} = B(W'_i)$. Therefore, Proposition 15 applies to $\{W'_i\}_{i\in\mathbb{Z}}$.■

Lemma 10: *Let* $\widetilde{V}$ and $\{W'_i\}_{i\in\mathbb{Z}_+}$ be defined as in Lemma 8. *Take* $\widetilde{W}_0 := W'_0$ *and let* $\widetilde{W}_{i+1} := \widetilde{B}\left(\widetilde{W}_i\right)$ *for* $\forall i \in \mathbb{Z}_+$. *Then,* $\widetilde{W}_i = W'_i$, $\forall i \in \mathbb{Z}_+$.

Proof: Assume $\widetilde{W}_{i-1} = W'_{i-1}$ for some $i \in \mathbb{Z}_{++}$. Then, $\widetilde{W}_{i-1} \subset W'_0$ by Lemma 8. Consequently, for $\forall {}^{-\theta}y \in Y^\theta$, $\widetilde{W}_i\left({}^{-\theta}y\right) \subset W'_i\left({}^{-\theta}y\right)$, i.e. $\widetilde{W}_i \subset W'_i$. Fix ${}^{-\theta}y \in Y^\theta$ and let $V \in W'_i\left({}^{-\theta}y\right)$. By Lemma 8, $W'_i \subset W'_{i-1} = \widetilde{W}_{i-1}$ from where $V \in \widetilde{W}_{i-1}\left({}^{-\theta}y\right)$. Then, $V \in \widetilde{B}_{{}^{-\theta}y}\left(\widetilde{W}_{i-1}\right)$. Since ${}^{-\theta}y$ and V were chosen randomly, we obtain $W'_i \subset \widetilde{W}_i$.

We have that $\widetilde{W}_0 = W'_0$ by definition and have just shown that $\widetilde{W}_{i-1} = W'_{i-1}$ would imply $\widetilde{W}_i = W'_i$; therefore, by induction we obtain that $\widetilde{W}_i = W'_i$, $\forall i \in \mathbb{Z}_+$.■

Proof of Proposition 16: From Lemmas 9 and 10.■

APPENDIX 2

Table 1
State Space of the Optimal ICAP Contract

Case		1		2		3	
		[	]	[	]	[	]
LLL		-3.1325	843.0178	-3.1325	29.4635	-3.1325	22.4035
	$y_{(1)}$	-1.4325	843.0178	-1.4325	29.4635	-1.4325	22.4035
LLM	$y_{(2)}$	-1.4325	843.0178	-1.4325	29.4635	-1.4325	22.4035
	$y_{(3)}$	0.0000	843.0178	0.0000	29.4635	0.0000	22.4035
	$y_{(1)}$	0.8075	843.0178	0.8056	29.4635	0.8050	22.4035
LLH	$y_{(2)}$	0.8075	843.0178	0.8056	29.4635	0.8050	22.4035
	$y_{(3)}$	3.6725	843.0178	3.6725	29.4635	3.6725	22.4030
	$y_{(1)}$	-0.1425	843.0178	-0.1460	29.4635	-0.1461	22.4035
LMM	$y_{(2)}$	0.0000	843.0178	0.0000	29.4635	0.0000	22.4035
	$y_{(3)}$	0.0000	843.0178	0.0000	29.4635	0.0000	22.4035
	$y_{(1)}$	0.8275	843.0178	0.8182	29.4635	0.8280	22.4035
LMH	$y_{(2)}$	0.8200	843.0178	0.8200	29.4635	0.8200	22.4035
	$y_{(3)}$	3.6725	843.0178	3.6725	29.4635	3.6725	22.4030
	$y_{(1)}$	3.3575	843.0178	3.3635	29.4635	3.3632	22.3724
LHH	$y_{(2)}$	3.6725	843.0178	3.6725	29.4635	3.6725	22.3783
	$y_{(3)}$	3.6725	843.0178	3.6725	29.4635	3.6725	22.3783
MMM		0.0000	843.0178	0.0000	29.4635	0.0000	22.4035
	$y_{(1)}$	0.8100	843.0178	0.8100	29.4635	0.8100	22.4035
MMH	$y_{(2)}$	0.8100	843.0178	0.8100	29.4635	0.8100	22.4035
	$y_{(3)}$	3.6725	843.0178	3.6725	29.4635	3.6725	22.4030
	$y_{(1)}$	3.3600	843.0178	3.3594	29.4635	3.3592	22.3623
MHH	$y_{(2)}$	3.6725	843.0178	3.6725	29.4635	3.6725	22.3703
	$y_{(3)}$	3.6725	843.0178	3.6725	29.4635	3.6725	22.3703
HHH		3.6725	843.0178	3.6725	29.4635	3.6725	22.4001

Table 2
Effects of Changing the Minimum Reservation Utility of the Principal
(LLL, case 1)

$\underline{U}$	0	5	10
$\overline{w}$	1145.5526	1010.7817	876.0108
$\widehat{V}$	843.0178	791.6873	736.8045
$\{V^{ICAP}\}$	[-3.1325, 843.0178]	[-3.1325, 791.6873]	[-3.1325, 736.8045]

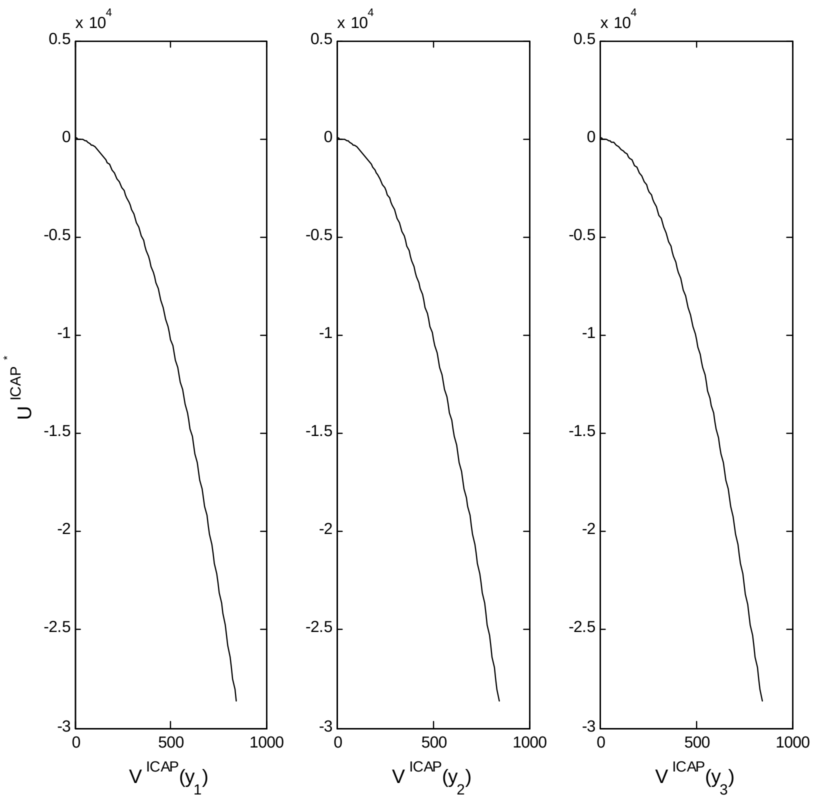

Figure 1: $U^{ICAP^*}(., y_i)$, $i \in \{1, 2, 3\}$ (LMH, case 1)

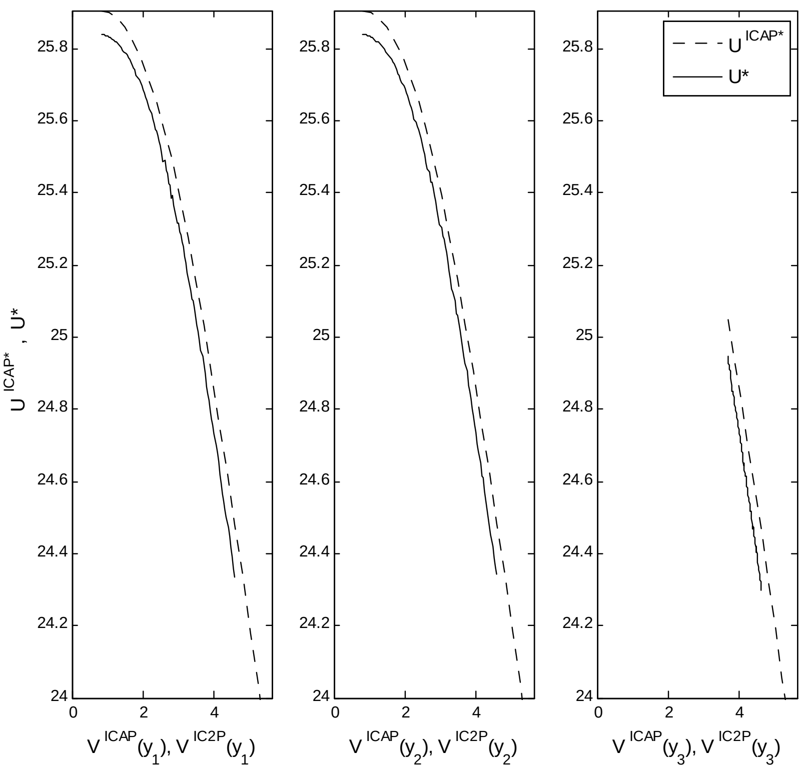

Figure 2: $U^{*}\left(.,y_{i}\right),\;\; U^{ICAP^{*}}\left(.,y_{i}\right),\;\; i\in\{1,2,3\}$ (LMH, case 1)

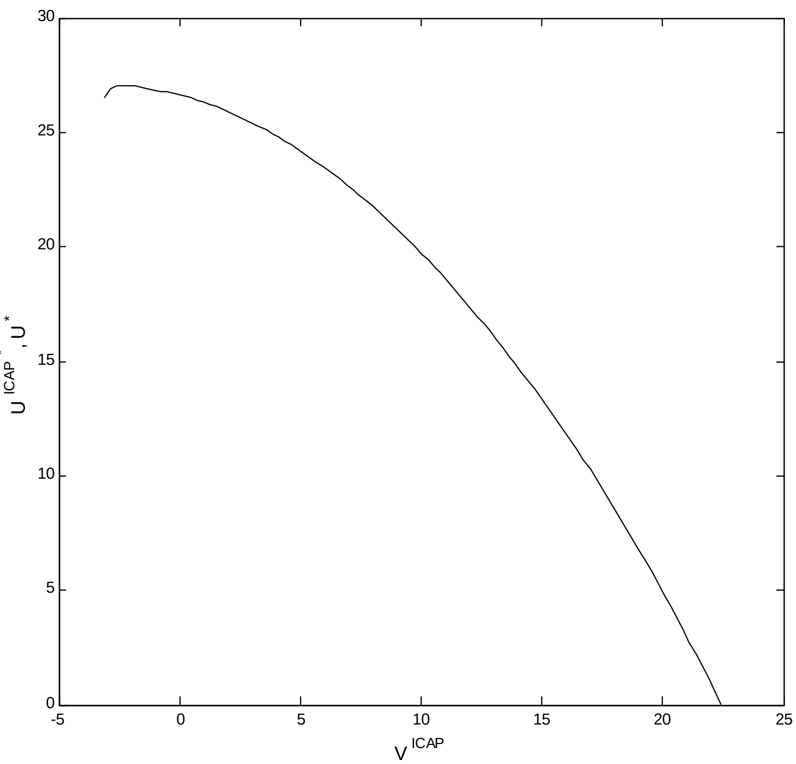

Figure 3: $U^{ICAP^*}(.)$, $U^*(.)$ (LLL, case 3)

Figure 4

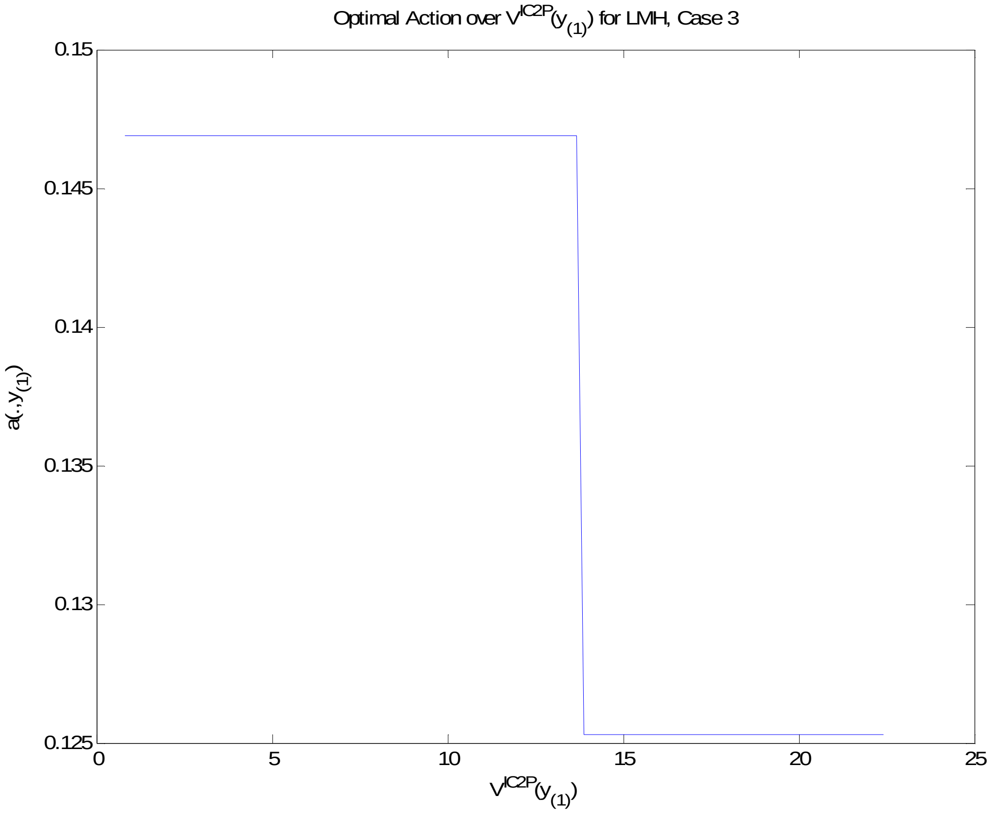
Optimal Action over V^IC2P(y_(1)) for LMH, Case 3
0.15
0.145
0.14
0.135
0.13
0.125
a(.,y_(1))
0
5
10
15
20
25
V^IC2P(y_(1))

Figure 5

Wage as a function of V given current price realization $y_{(1)}$, initial history $y_{(1)}$, LMH, Case 3

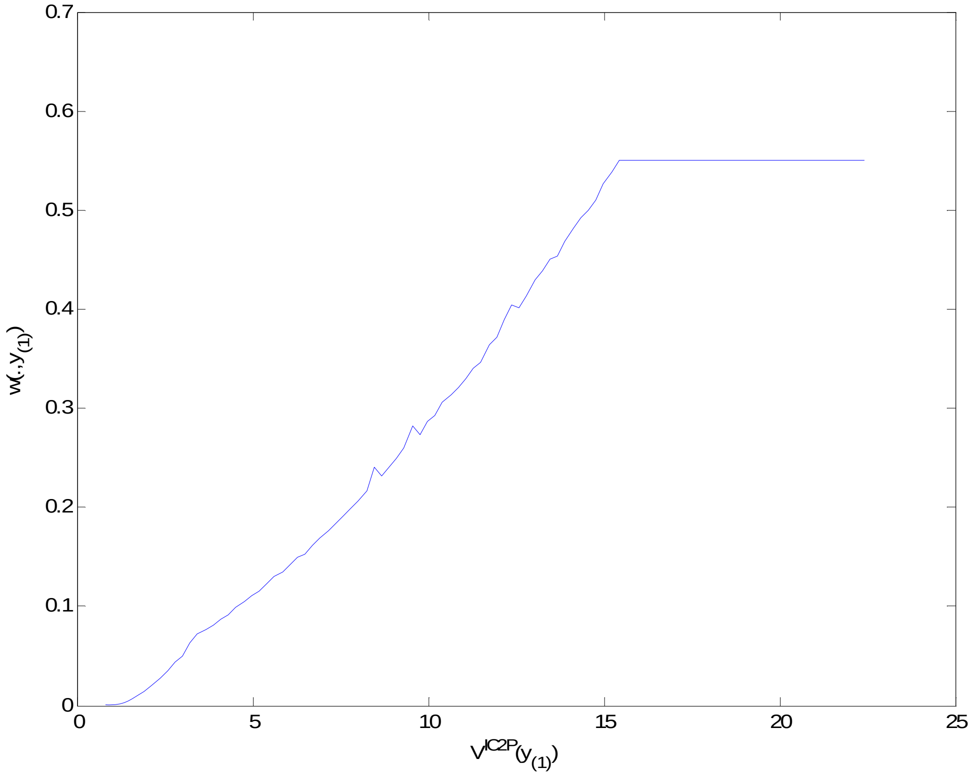

Figure 6

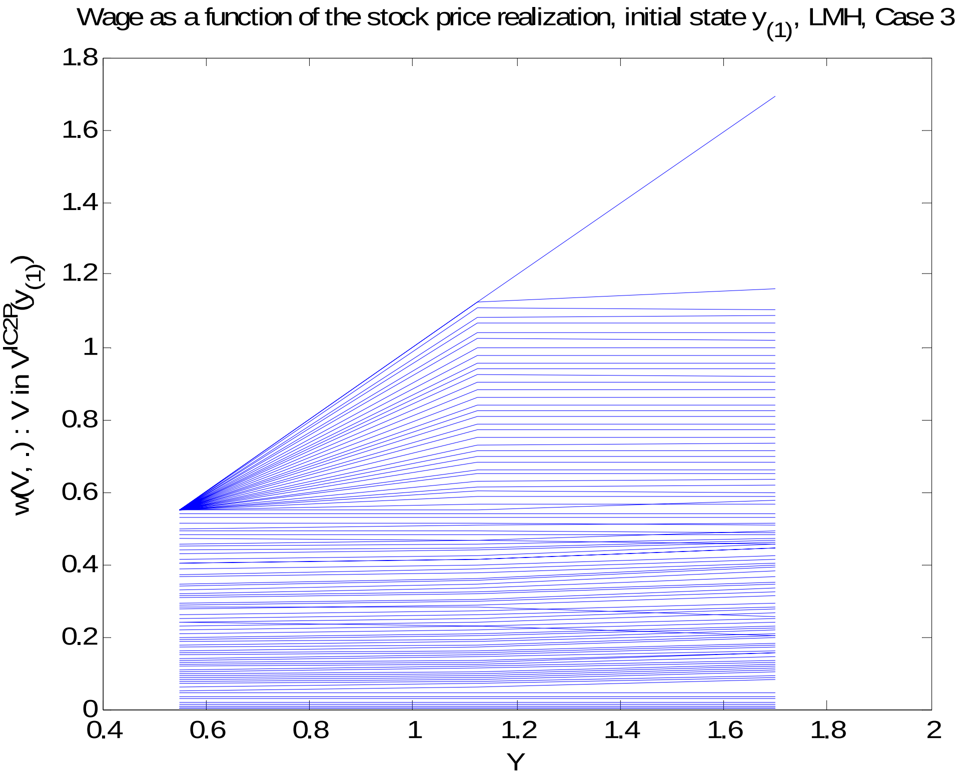
Wage as a function of the stock price realization, initial state y(1), LMH, Case 3
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Y
w(V, .) : V in V^IC2P(y(1))

Figure 7

Tomorrow's utility as a function of V given current price realization $y_{(1)}$, initial history $y_{(1)}$, LMH, Case 3

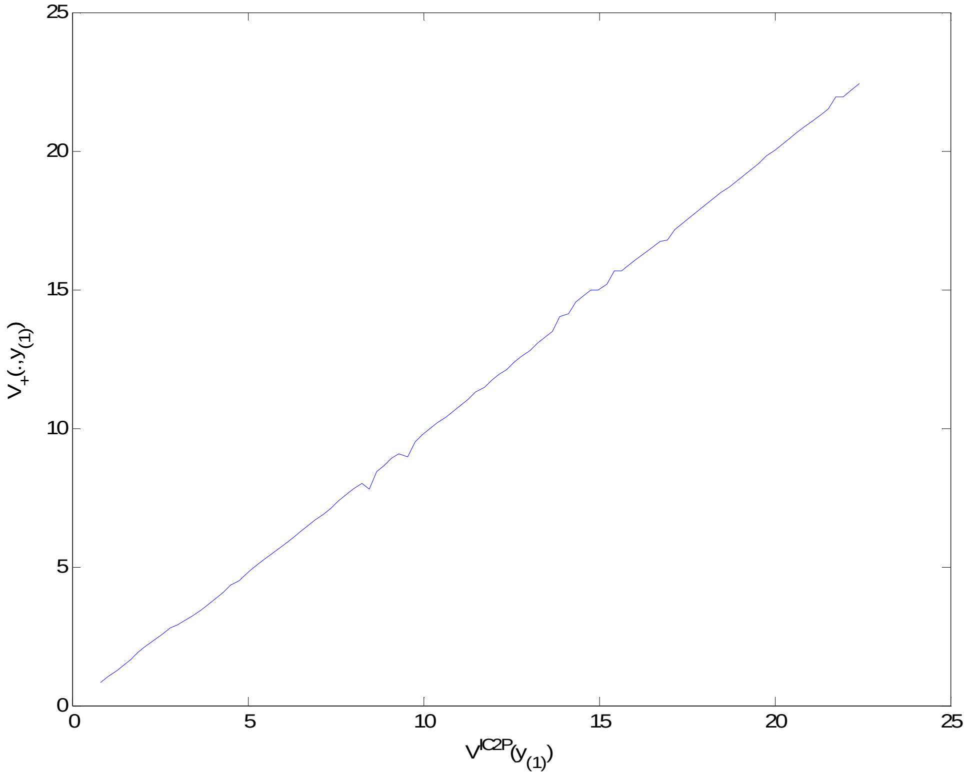

Figure 8

Tomorrow's utility as a function of the stock price realization, initial history $y_{(1)}$, LMH, Case 3

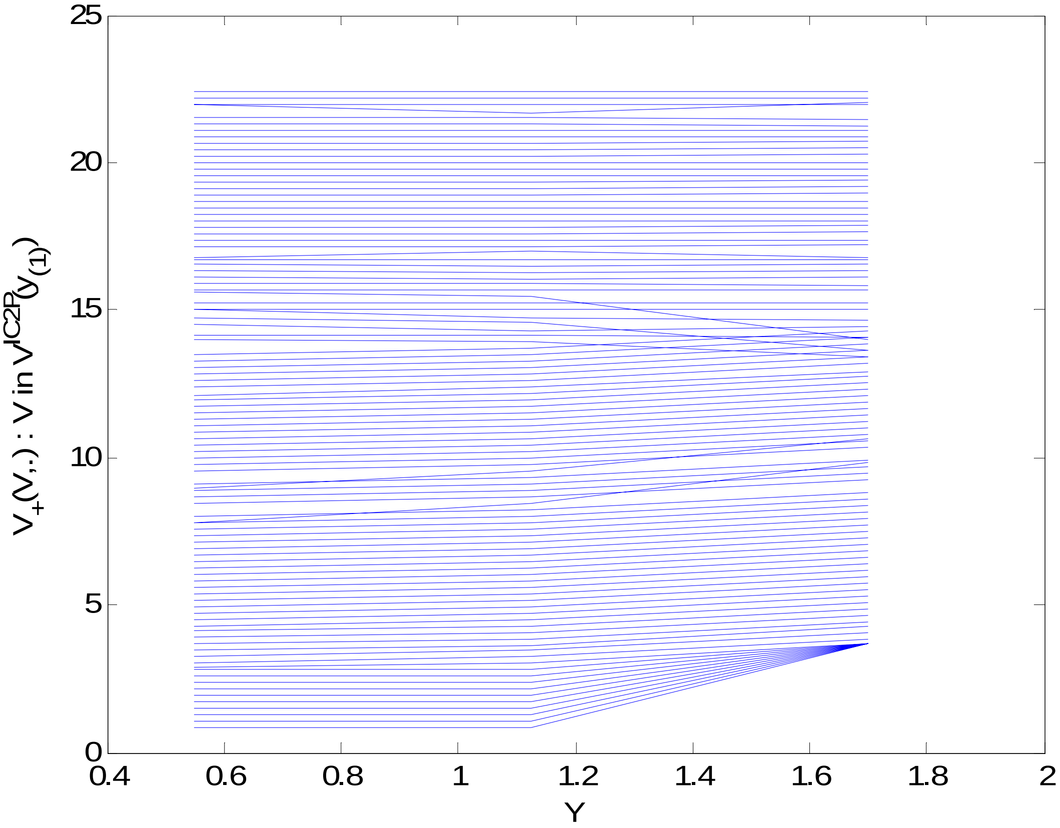

Figure 9
Agent's utility in time given a starting point $(y_{(1)},V_0)$ s.t. V_0 in $V^{IC2P}(y_{(1)})$, LMH, Case 3