COMPARING BERTRAND AND COURNOT OUTCOMES IN THE PRESENCE OF PUBLIC FIRMS

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ABSTRACT. This paper characterizes and compares Bertrand and Cournot outcomes in a differentiated duopoly in which a public firm competes against a private firm. The equilibrium price of a welfare-maximizing public firm is strictly lower in Cournot than in Bertrand competition. Contrary to the Bertrand-Cournot ranking in standard comparison (with profit-maximizing firms only), profits in Cournot are strictly lower and consumer surplus is strictly higher under a linear demand structure. All these results hold with more than two firms under a range of parameter values. The reversals also hold in a richer setting with a partially privatized public firm, where the extent of privatization is endogenously determined by a welfare-maximizing government. As a by-product of our analysis, we find that in a differentiated duopoly setting, partial privatization always improves welfare in Cournot but not necessarily in Bertrand competition.

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1. Introduction

It is now well known that Bertrand competition yields lower prices and profits and higher consumer surplus and welfare than Cournot competition (see, for example, Singh and Vives [17], Cheng [4], Vives [20], and Okuguchi [14]). Subsequently, exploiting cost asymmetries, Dastidar [5], Qiu [15], and Häckner [11] constructed specific examples where at least one of these conclusions fails to hold. However, to date, the literature comparing Bertrand and Cournot has almost exclusively focused on environments where all firms maximize profits. We revisit the classic comparison, in *mixed markets*, where profit-maximizing private firms coexist with public firms.

In developed countries, this coexistence is observed in several oligopolistic sectors including banking, insurance, and telecommunications. In the developing world, the share of public enterprises in manufacturing output and employment lies in the 30-70% range (see Schmitz [16]). The following excerpt from World Bank Report (1995) captures the pervasiveness of public firms in developing countries:

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"Government employees operate a casino in Ghana; bake cookies in Egypt; assemble watches in India; mine salt in Mexico; make matches in Mali, and bottle cooking oil in Senegal."

So, how does a public firm differ from its private counterpart? Following the mixed oligopoly literature, we assume that the difference lies in the objective function. Unlike private firms, which maximize profits, a public firm maximizes welfare (sum of consumer surplus and producer surplus).[1] We characterize and compare Cournot and Bertrand outcomes in a differentiated duopoly where a private firm competes against a welfare-maximizing public firm (sections 3 and 4). The results are strikingly different from the ones obtained from a similar comparison with profit-maximizing firms only. The standard Bertrand-Cournot rankings are reversed for prices, consumer surplus, and profits. These reversal results generalize beyond duopoly for a wide range of parameter values.

In section 5 we re-examine the comparison between Bertrand and Cournot outcomes in presence of partial privatization. To endogenize the degree of privatization we construct a stylized two-stage game in which a public firm maximizes a weighted sum of its own profits and welfare in stage two, and the weights, indicating the extent of privatization, are chosen optimally by a welfare-maximizing government in stage one. The reversals also hold under this richer setting. As a by-product of our analysis, we find that the mode of market competition (i.e., Bertrand or Cournot) can have qualitatively different welfare implications for partial privatization.

2. A Preview of Results

Subsections 2.1 to 2.3 preview the results from section 4 while the remaining subsections discuss the results from section 5. Unless otherwise mentioned, the results pertain to the duopoly case. Robustness of the results with more than two firms are explored in Remarks 1-3.

2.1. **Lower prices under Cournot.** A welfare-maximizing public firm sets a strictly lower price in Cournot competition than in Bertrand competition (Proposition 1). The result holds irrespective of the demand specification. Under the linear demand structure (see (6) and (7)), the predominant case considered in the literature, a private firm's price is the same under Bertrand and Cournot in a mixed duopoly (Proposition 3 (i)). If there are more than two firms of each type (public and private), all prices, public as well as private, are strictly lower in Cournot for a range of parameterizations provided the products are not close substitutes (Remark 2).

In a homogenous product oligopoly with strictly convex and asymmetric costs, Dastidar [5] has shown that the prices can be lower in Cournot competition under some sharing rules. In a differentiated oligopoly with more than two firms, Qiu [15], Häckner [11], and Amir and Jin [1] have shown that Cournot equilibrium prices might be lower for some firms, provided costs are asymmetric. Our result complements this literature by showing

[1]See De Fraja and Delbono [7] and Basu [3, Ch.16] for surveys on mixed oligopoly. Also, see De Fraja and Delbono [6] for a critical discussion on welfare maximization as an objective of public firms.

that even with symmetric costs, we can have lower prices under Cournot competition if there is a public firm.

2.2. **Lower profits and higher consumer surplus in Cournot.** Lower Cournot prices under linear demand structure give rise to (i) higher consumer surplus and (ii) lower profits under Cournot competition (Proposition 4 (i) and (ii)). Thus, compared to the standard framework with profit-maximizing firms only, we have a reversal of Bertrand-Cournot ranking for consumer surplus and profits. These reversals hold irrespective of the degree of differentiation in a duopoly. In an oligopoly, these reversals hold for a range of parameterizations.

2.3. **Higher welfare in Cournot?** Despite the reversals in the Bertrand-Cournot ranking for consumer surplus and profits, welfare remains strictly higher under Bertrand. Under general demand function, as long as a private firm's price is weakly lower in Bertrand competition, welfare reversal does not occur. In an oligopoly with more than two firms, there are parameterizations where private prices are strictly higher in Bertrand and furthermore, aggregate output is higher in Cournot. However, even for those parameterizations, welfare reversal does not occur.

In section 5, we reconsider the comparison between Bertrand and Cournot with a partially privatized public firm, which maximizes a weighted sum of its own profits and welfare.[2] Recognizing that incentives for privatization might be different under Bertrand and Cournot, we endogenize the extent of privatization. Adapting the models from the strategic delegation literature (see, for example, Vickers [19], Fershtman and Judd [9], and Sklivas [18]), we construct a stylized two-stage game in which a public firm maximizes a weighted sum of its own profits and welfare in stage two, and the weight attached to profits, capturing the extent of privatization, is chosen optimally by a welfare-maximizing government in stage one.

2.4. **Privatization and the mode of competition.** Irrespective of the demand specification, the optimal weight on profits of the public firm is strictly positive under Cournot competition (Proposition 5) but not necessarily so under Bertrand competition (Propositions 6 and 7). That is, partial privatization improves welfare in Cournot irrespective of the demand specification while it can reduce welfare in Bertrand competition.

2.5. **Bertrand versus Cournot with endogenous privatization.** The ranking reversals obtained in the basic model, i.e. lower profits and higher consumer surplus under Cournot, continue to hold in this richer setting with endogenous privatization as long as the products are not very close substitutes (Proposition 8).

[2]See Fershtman [10] and Matsumura [13] for similar modelling of partial privatization.

3. The Basic Model

Consider an economy with two sectors: a competitive sector producing the numeraire good y and an imperfectly competitive sector with two firms, firm 1 and firm 2, each one producing a differentiated good. Let p_i and q_i respectively denote firm i's price and quantity where $i = 1, 2$. The representative consumer maximizes $V(q, y) \equiv U(q) + y$ subject to $p_1q_1 + p_2q_2 + y \leq I$ where $q \equiv (q_1, q_2) \in \Re^2_+$ and I denotes income. The utility function $U(q)$ is continuously differentiable as often as is required on $\Re^2_+$. Furthermore the following holds:

Assumption 1. For $i, j \in \{1, 2\}$, $i \neq j$, (i) $U_i(q) \equiv \frac{\partial U(q)}{\partial q_i} > 0$, (ii) $U_{ii}(q) \equiv \frac{\partial^2 U(q)}{\partial {q_i}^2} < 0$, (iii) $U_{ij}(q) \equiv \frac{\partial^2 U(q)}{\partial q_i \partial q_j} < 0$, (iv) $U_{ij}(q) = U_{ji}(q)$ and (v) $|U_{ii}(q)| > |U_{ij}(q)|$.

These assumptions are standard in the literature (see, for example, section 5 in Singh or Vives [17]).

Since $V(q, y)$ is separable and linear in y, there are no income effects and consequently, for a large enough income, the representative consumer's optimization problem is reduced to choosing q to maximize $U(q) - p_1q_1 - p_2q_2$. Utility maximization yields the inverse demands: $p_i = \frac{\partial U(q)}{\partial q_i} \equiv P_i(q)$ for $q_i > 0$, $i = 1, 2$. Then, applying Assumption 1 gives two important properties: (a) demand slopes downward, since $\frac{\partial P_i(q)}{\partial q_i} \equiv U_{ii}(q) < 0$, and (b) two goods are substitutes, since $\frac{\partial P_i(q)}{\partial q_j} \equiv U_{ij}(q) < 0$, $(i \neq j)$.

Inverting the inverse demand system yields the direct demands: $q_i = D_i(p)$ where $i = 1, 2$ and $p = (p_1, p_2) \in \Re^2_+$. From the assumptions on $U(q)$, we have

$$\frac{\partial D_i(p)}{\partial p_i} = \frac{U_{jj}}{U_{11}U_{22} - U_{12}U_{21}} < 0, \frac{\partial D_i(p)}{\partial p_j} = -\frac{U_{ij}}{U_{11}U_{22} - U_{12}U_{21}} > 0, \tag{1}$$

where $i, j \in \{1, 2\}$, $i \neq j$ and U_{ij}, U_{ii} are evaluated at $q = (q_1, q_2) \equiv (D_1(p), D_2(p))$.

Each firm has a constant marginal cost $m > 0$.[3] There are no fixed costs. Firm 2 maximizes profit and we refer to firm 2 as the private firm. Firm 1 is a public sector enterprise, or, in short, a public firm. Following the mixed oligopoly literature, we assume that the public firm maximizes welfare. Profits and welfare, in terms of q and p, are precisely defined below.

3.1. Cournot competition.

Corresponding to a quantity vector $q \equiv (q_1, q_2)$, profits of firm i, denoted by $\pi_i(q)$, and welfare, denoted by $W(q)$, are:

$$\begin{aligned} \pi_i(q) &= (p_i(q) - m)q_i, \\ W(q) &= U(q) - m(q_1 + q_2). \end{aligned}$$

We assume that q_1 and q_2 are strategic substitutes. More formally, the following holds:

[3] We assume that unit costs are constant and symmetric for the public and private firm to highlight on our source of reversal — the presence of public firms. Qualitatively, all our propositions hold under asymmetric costs.

Assumption 2. For $i, j \in \{1,2\}$, $i \neq j$, we have (i) $\frac{\partial^2 W}{\partial q_i \partial q_j} < 0$, and (ii) $\frac{\partial^2 \pi_i}{\partial q_i \partial q_j} < 0$.

In a Cournot oligopoly with profit-maximizing firms, quantities are usually assumed to be strategic substitutes. Assumption 2 says that strategic substitutability holds even when one firm maximizes welfare.

A quantity vector $q^C \equiv (q_1^C, q_2^C)$ is a Cournot equilibrium if and only if $W(q^C) \geq W(q_1, q_2^C)$ for all $q_1 \neq q_1^C$ and $\pi_2(q^C) \geq \pi_2(q_1^C, q_2)$ for all $q_2 \neq q_2^C$. Corresponding to a Cournot equilibrium quantity vector q^C, define

$$\begin{aligned} p_i^C &= P_i(q^C), \\ \pi_i^C &= \pi_i(q^C), \\ CS^C &= U(q^C) - p_1^C q_1^C - p_2^C q_2^C, \quad \text{and} \\ W^C &= W(q^C) = U(q^C) - m(q_1^C + q_2^C), \end{aligned}$$

where p_i^C, π_i^C, CS^C, and W^C respectively are firm i's price, firm i's profits, consumer surplus, and welfare in Cournot equilibrium.

3.2. **Bertrand competition.** Corresponding to a price vector $p \equiv (p_1, p_2)$, profits of firm i, denoted by $\tilde{\pi}_i(p)$, and welfare, denoted by $\tilde{W}(p)$, are:

$$\begin{aligned} \tilde{\pi}_i(p) &= (p_i - m) D_i(p), \\ \tilde{W}(p) &= U(D_1(p), D_2(p)) - m(D_1(p) + D_2(p)). \end{aligned}$$

We assume that p_1 and p_2 are strategic complements. More formally, the following holds:

Assumption 3. For $i, j \in \{1,2\}$, $i \neq j$, we have (i) $\frac{\partial^2 W}{\partial p_i \partial p_j} > 0$, and (ii) $\frac{\partial^2 \pi_i}{\partial p_i \partial p_j} < 0$.

Strategic complementarity (of prices) is a standard assumption in a differentiated products Bertrand oligopoly with profit-maximizing firms. Assumption 2 says that strategic complementarity holds even when one firm maximizes welfare.

A price vector $p^B \equiv (p_1^B, p_2^B)$ is a Bertrand equilibrium if and only if $\tilde{W}(p^B) \geq \tilde{W}(p_1, p_2^B)$ for all $p_1 \neq p_1^B$ and $\tilde{\pi}_2(p^B) \geq \tilde{\pi}_2(p_1^B, p_2)$ for all $p_2 \neq p_2^B$. Corresponding to a Bertrand equilibrium price vector p^B, define

$$\begin{aligned} q_i^B &= D_i(p^B), \\ \pi_i^B &= \tilde{\pi}_i(p^B) \equiv \pi_i(q^B), \\ CS^B &= U(q^B) - p_1^B q_1^B - p_2^B q_2^B, \\ W^B &= \tilde{W}(p^B) \equiv W(q^B) = U(q^B) - m(q_1^B + q_2^B), \end{aligned}$$

where q_i^B, π_i^B, CS^B, and W^B respectively are firm i's price, firm i's profits, consumer surplus, and welfare in Bertrand equilibrium.

4. Comparing Bertrand and Cournot Outcomes

First we compare the pricing of the public firm under Cournot and Bertrand competition. In a differentiated duopoly with profit-maximizing firms only, equilibrium prices are higher under Cournot competition (Singh and Vives [17]; Vives [20]). Proposition 1 shows that the conclusion does not hold in the presence of a public firm. The public firm's price is *strictly lower* under Cournot competition irrespective of demand specification.

Proposition 1. Suppose $q_i^{\mathcal{C}} > 0$ and $q_i^{\mathcal{B}} > 0$ for $i = 1, 2$. Then $p_1^{\mathcal{B}} > p_1^{\mathcal{C}} = m$.

Proof: Under Cournot competition, $q_1 = q_1^C$ maximizes $W(q_1, q_2^{\mathcal{C}})$. Since $q_i^C > 0$, from first order conditions we get $\frac{\partial W(q^C)}{\partial q_1} = \frac{\partial U(q\mathcal{C})}{\partial q_1} - m = 0$. We have $\frac{\partial U(q^C)}{\partial q_1} \equiv P_1(q^C)$ and by definition, $p_1^C = P_1(q^C)$. Thus $p_1^{\mathcal{C}} = m$.

Noting that $p^B = (p_1^B, p_2^B)$ constitutes the Bertrand equilibrium, from first order conditions we get:

$$\frac{\partial \tilde{W}(p^{\mathcal{B}})}{\partial p_1} = (p_1^{\mathcal{B}} - m)\frac{\partial D_1(p^{\mathcal{B}})}{\partial p_1} + (p_2^{\mathcal{B}} - m)\frac{\partial D_2(p^{\mathcal{B}})}{\partial p_1} = 0, \tag{2}$$

$$\frac{\partial \tilde{\pi}_2(p^{\mathcal{B}})}{\partial p_2} = (p_2^{\mathcal{B}} - m)\frac{\partial D_2(p^{\mathcal{B}})}{\partial p_2} + D_2(p^{\mathcal{B}}) = 0. \tag{3}$$

From (1), we have (i) $\frac{\partial D_1(p^{\mathcal{B}})}{\partial p_1} < 0$ and (ii) $\frac{\partial D_2(p^{\mathcal{B}})}{\partial p_1} > 0$. Since $\frac{\partial D_2(p^{\mathcal{B}})}{\partial p_2} < 0$ and $D_2(p^{\mathcal{B}}) = q_2^B > 0$, from (3) it follows that (iii) $p_2^{\mathcal{B}} - m > 0$. Together with (i)-(iii), Equation (2) implies that $p_1^{\mathcal{B}} - m > 0$. □

To understand Proposition 1, consider an infinitesimally small increase in p_1 from $p_1 = m$. This reduces the public firm's output, q_1, and raises the private firm's output, q_2. The welfare loss from reduction in q_1 is second order while the welfare gain from increase in q_2 is first order(since $p_2^{\mathcal{B}} > m$). This logic implies that if the two goods are substitutes, and firms compete in prices, a welfare-maximizing public firm will set its price strictly higher than marginal cost. Thus $p_1^B > m$. Such considerations are absent in Cournot competition. Under Cournot conjecture, the public firm chooses its own output, taking the private firm's output as given. Thus public firm behaves like a welfare-maximizing monopolist, which in turn yields $p_1^{\mathcal{C}} \equiv \frac{\partial U(q^{\mathcal{C}})}{\partial q_1} = m$.

Though the public firm's price is strictly lower, the private firm's price can be lower or higher under Cournot depending on the utility/demand specification. Before comparing the private firm's prices under Bertrand and Cournot, we provide a characterization result that links the private firm's price and welfare for any utility specification satisfying Assumption 1.

Proposition 2. Suppose $p_2^{\mathcal{C}} \geq p_2^{\mathcal{B}}$. Then $W^{\mathcal{B}} > W^{\mathcal{C}}$.

Proof: Noting that $W^{\mathcal{B}} \equiv \tilde{W}(p^{\mathcal{B}})$ and $W^{\mathcal{C}} \equiv W(q^{\mathcal{C}}) = \tilde{W}(p^{\mathcal{C}})$ we get:

$$\begin{aligned} W^{\mathcal{B}} - W^{\mathcal{C}} &= \tilde{W}(p^{\mathcal{B}}) - \tilde{W}(p_1^{\mathcal{C}}, p_2^{\mathcal{B}}) + \tilde{W}(p_1^{\mathcal{C}}, p_2^{\mathcal{B}}) - \tilde{W}(p^{\mathcal{C}}), \\ &= \tilde{W}(p^{\mathcal{B}}) - \tilde{W}(p_1^{\mathcal{C}}, p_2^{\mathcal{B}}) + \int_{p_2^{\mathcal{C}}}^{p_2^{\mathcal{B}}} \frac{\partial \tilde{W}(p_1^{\mathcal{C}}, p_2)}{\partial p_2} dp_2. \end{aligned} \tag{4}$$

Since (i) $p^{\mathcal{B}}$ constitutes Bertrand equilibrium, and (ii) $p_1^{\mathcal{C}} = m$ is not a best response to $p_2^{\mathcal{B}}$, [4] we have $\tilde{W}(p^{\mathcal{B}}) - \tilde{W}(p_1^{\mathcal{C}}, p_2^{\mathcal{B}}) > 0$.

[4] This follows from noting that $\frac{\partial \tilde{W}(p_1^{\mathcal{C}}, p_2^{\mathcal{B}})}{\partial p_1} = (p_2^{\mathcal{B}} - m)\frac{\partial D_2(p_1^{\mathcal{C}}, p_2^{\mathcal{B}})}{\partial p_1} > 0$.

Then, given $p_2^{\mathcal{C}} \geq p_2^{\mathcal{B}}$ it suffices to show that $\frac{\partial \tilde{W}(p_1^{\mathcal{C}}, p_2)}{\partial p_2} < 0$ for all $p_2 \in (p_2^{\mathcal{B}}, p_2^{\mathcal{C}})$. We have

$$\begin{aligned} \frac{\partial \tilde{W}(p_1^{\mathcal{C}}, p_2)}{\partial p_2} &= (p_1^{\mathcal{C}} - m)\frac{\partial D_1(p_1^{\mathcal{C}}, p_2)}{\partial p_2} + (p_2 - m)\frac{\partial D_2(p_1^{\mathcal{C}}, p_2)}{\partial p_2}, \\ &= (p_2 - m)\frac{\partial D_2(p_1^{\mathcal{C}}, p_2)}{\partial p_2} \quad \text{(note } p_1^{\mathcal{C}} = m.) \end{aligned}$$

Clearly $p_2^{\mathcal{B}} > m$ and $p_2^{\mathcal{C}} > m$. Consequently (i) $p_2 - m > 0$ for all $p_2 \in [p_2^{\mathcal{B}}, p_2^{\mathcal{C}}]$. Also, by (1), we have (ii) $\frac{\partial D_2(p_1^{\mathcal{C}}, p_2)}{\partial p_2} < 0$. Together, (i) and (ii) imply $\frac{\partial \tilde{W}(p_1^{\mathcal{C}}, p_2)}{\partial p_2} < 0$ for all $p_2 \in [p_2^{\mathcal{B}}, p_2^{\mathcal{C}}]$. □

Proposition 2 states that if the private firm's Cournot price is higher than or equal to the Bertrand price, then welfare reversal cannot occur. This is a strong result since one would expect that if $p_1^{\mathcal{C}} < p_1^{\mathcal{B}}$ (as in Proposition 1) and $p_2^{\mathcal{C}} \geq p_2^{\mathcal{B}}$, then there is a possibility of welfare reversal. Proposition 2 rules out such a possibility.

Remark 1. Does Proposition 2 generalize beyond duopoly? The short answer is yes. To see this, consider a $n(\geq 2)$-firm oligopoly with n_1 public firms and $n_2(\equiv n - n_1)$ private firms. Assume $1 \leq n_1 < n$. Label these firms from 1 to n such that firms labeled 1 to n_1 are public while firms labeled $n_1 + 1$ to n are private. Let $G_1 = \{1, 2, ..., n_1\}$ and $G_2 = \{n_1 + 1, n_1 + 2, ..., n\}$ denote the group of public firms and the group of private firms respectively. Assume an interior Bertrand equilibrium exists. Then there exists a Bertrand equilibrium with the following property: $p_i^{\mathcal{B}} = p_1^{\mathcal{B}}$ if $i \in G_1$, and $p_i^{\mathcal{B}} = p_2^{\mathcal{B}}$ if $i \in G_2$. Similarly, an interior Cournot equilibrium exists in which the following holds: $p_i^{\mathcal{C}} = m$ if $i \in G_1$, and $p_i^{\mathcal{C}} = p_2^{\mathcal{C}}$ if $i \in G_2$. Then we find: $p_2^{\mathcal{C}} \geq p_2^{\mathcal{B}} \Rightarrow W^{\mathcal{B}} > W^{\mathcal{C}}$ provided $\sum_{j \in G_2} \frac{\partial D_i(\mathbf{m}, \mathbf{p})}{\partial p_j} < 0$ for all $i \in G_2$, where $\mathbf{m}$ is a n_1-element vector with all elements m and $\mathbf{p}$ is a n_2-element vector with all elements p. Similar conditions are invoked in differentiated oligopoly literature to ensure that own-price effects dominate the cross-price effects. See, for example, condition (A.3) in Vives [20] or pp. 157 in Vives [21].

The lower prices of a public firm under Cournot (Proposition 1) opens up the possibility of reversal of Bertrand-Cournot orderings for a private firm's price, quantities, consumer surplus, profits, and welfare. However, to determine whether reversals actually occur or not, we need to specify utility function. We consider the quadratic utility specification proposed in Dixit [8] and subsequently used in Singh and Vives [17], Qiu [15], Häckner [11] and several other papers in this literature:

$$U(q) = a(q_1 + q_2) - \frac{1}{2}(q_1^2 + q_1^2) - bq_1q_2, \tag{5}$$

where $a > c$, and $b \in (0, 1)$. The goods are independent if $b = 0$ and perfect substitutes if $b = 1$. The restriction that b lie strictly between 0 and 1 implies that the goods are imperfect substitutes. The degree of substitutability increases, or equivalently, the extent of product differentiation declines, as

b increases. It is easy to verify that utility specification in (5) satisfies Assumption 1.

The inverse demands corresponding to (5) are linear and given by

$$p_1 = a - q_1 - bq_2, \quad p_2 = a - q_2 - bq_1. \tag{6}$$

Inverting the inverse demands yields the direct demands:

$$q_1 = \frac{a(1-b) - p_1 + bp_2}{1-b^2}, \quad q_2 = \frac{a(1-b) - p_2 + bp_1}{1-b^2}. \tag{7}$$

For this linear demand system, Cournot competition yields strictly higher prices and a strictly lower output compared to Bertrand competition in a standard differentiated duopoly (Singh and Vives [17]). Proposition 3 shows that except for the private firm's output, none of these conclusions hold.

Proposition 3. Suppose $U(q)$ is given by (5). Then
(i) $p_1^{\mathcal{C}} < p_1^{\mathcal{B}}$, $p_2^{\mathcal{C}} = p_2^{\mathcal{B}}$;
(ii) $q_1^{\mathcal{C}} > q_1^{\mathcal{B}}$, $q_2^{\mathcal{C}} < q_2^{\mathcal{B}}$.

Proof: If $U(q)$ is given by (5), then under Cournot competition, the following first order conditions hold at $(q_1, q_2) = (q_1^{\mathcal{C}}, q_2^{\mathcal{C}})$:

$$\frac{\partial W}{\partial q_1} = a - q_1 - bq_2 - m = 0, \quad \frac{\partial \pi_2}{\partial q_2} = a - 2q_2 - bq_1 - m = 0.$$

Note, $\frac{\partial^2 W}{\partial q_1 \partial q_2} = \frac{\partial^2 \pi_2}{\partial q_1 \partial q_2} = -b < 0$. Thus Assumptions 2 (i) and 2 (ii) hold. Solving the two first order conditions gives

$$q_1^{\mathcal{C}} = \frac{(2-b)(a-m)}{2-b^2}, \quad q_2^{\mathcal{C}} = \frac{(1-b)(a-m)}{2-b^2}.$$

Substituting q_i by $q_i^{\mathcal{C}}$, $i = 1, 2$ in (7) yields equilibrium prices under Cournot:

$$p_1^{\mathcal{C}} = m, \quad p_2^{\mathcal{C}} = m + \frac{(1-b)(a-m)}{2-b^2}.$$

The first order conditions under Bertrand competition are given by (2) and (3). If $U(q)$ is given by (5), $\frac{\partial^2 \tilde{W}}{\partial p_1 \partial p_2} = \frac{\partial^2 \pi_2}{\partial p_1 \partial p_2} = \frac{b}{1-b^2} > 0$. Thus Assumptions 3 (i) and 3 (ii) hold. Substituting $\frac{\partial D_i}{\partial p_i} = \frac{-1}{1-b^2}$ and $\frac{\partial D_i}{\partial p_j} = \frac{b}{1-b^2}$ $(i \neq j)$ and then solving (2) and (3) gives

$$p_1^{\mathcal{B}} = m + \frac{b(1-b)(a-m)}{2-b^2}, \quad p_2^{\mathcal{B}} = m + \frac{(1-b)(a-m)}{2-b^2}.$$

Substituting p_i by $p_i^{\mathcal{B}}$, $i = 1, 2$ in (6) yields equilibrium quantities under Bertrand:

$$q_1^{\mathcal{B}} = \frac{a-m}{1+b} \quad q_2^{\mathcal{B}} = \frac{(a-m)}{(1+b)(2-b^2)}.$$

Comparing $p_i^{\mathcal{C}}$ and $p_i^{\mathcal{B}}$ gives (i). Similarly, comparing $q_i^{\mathcal{C}}$ and $q_i^{\mathcal{B}}$ gives (ii). $\square$

From Proposition 1 we already know that $p_1^{\mathcal{C}} < p_1^{\mathcal{B}}$. Proposition 3 (i) says that $p_2^{\mathcal{C}} = p_2^{\mathcal{B}}$. Thus both prices are lower in Cournot. This is in sharp contrast to the standard ordering where equilibrium prices are strictly higher in Cournot.

Needless to say, the equality, that is, $p_2^{\mathcal{C}} = p_2^{\mathcal{B}}$, is unlikely to hold for an arbitrary number of firms.[5] Underlying the equality, however, are two opposing effects which are quite general. On the one hand, as in the standard setting, the perceived elasticity of demand of a firm is smaller under Cournot conjecture which raises $p_2^{\mathcal{C}}$.[6] On the other hand, a lower $p_1^{\mathcal{C}}$ (compared to $p_1^{\mathcal{B}}$) in our framework creates a downward pressure on the private firm's price which lowers $p_2^{\mathcal{C}}$. When $U(q)$ is given by (5) these two effects offset each other, which in turn yields $p_2^{\mathcal{C}} = p_2^{\mathcal{B}}$.

For $n > 2$, we find a wide range of parameter values for which $p_2^{\mathcal{C}} < p_2^{\mathcal{B}}$. Together with Proposition 1, this finding implies that for a range of parameter values, all prices, public as well as private, are strictly lower under Cournot. See Remark 2 for details.

Remark 2. Consider an n-firm oligopoly with n_1 welfare-maximizing public firms and $n_2 \equiv n - n_1$ private firms. Then generalize the utility function as follows:

$$U(q) = a\sum_{i=1}^{n} q_i - \frac{1}{2}\sum_{i=1}^{n} q_i^2 - b\sum_{i}\sum_{j<i} q_i q_j, \tag{8}$$

where $a > m$ and $b \in (0, 1)$ and each firm $i = 1, 2, ..., n$ produces exactly one variety. The equilibrium prices in Bertrand and Cournot exhibit within-group symmetry as mentioned in Remark 1. Let $p_1^{\mathcal{C}}(p_1^{\mathcal{B}})$ and $p_2^{\mathcal{C}}(p_2^{\mathcal{B}})$ denote the equilibrium prices charged by a public firm and a private firm respectively in Cournot (Bertrand) competition. Similarly, let $q_1^{\mathcal{C}}(q_1^{\mathcal{B}})$ and $q_2^{\mathcal{C}}(q_2^{\mathcal{B}})$ denote the equilibrium quantities produced by the public and private firm respectively in Cournot (Bertrand). We find:

(a) $p_1^{\mathcal{C}} = m < p_1^{\mathcal{B}}$, $\text{sgn}[p_2^{\mathcal{C}} - p_2^{\mathcal{B}}] = \text{sgn}[(n_2 - 1)(b(n-1) - (n_1 - 1))]$,[7]
(b) $q_1^{\mathcal{C}} > q_1^{\mathcal{B}}$, $q_2^{\mathcal{C}} < q_2^{\mathcal{B}}$.

The output comparisons are the same as in Proposition 3 (ii). Regarding price comparison, note that we get $p_2^{\mathcal{C}} = p_2^{\mathcal{B}}$ as in Proposition 3 (i), if $n_2 = 1$. If $n_2 > 1$, $p_2^{\mathcal{C}} < p_2^{\mathcal{B}}$ for $b \in (0, \frac{n_1-1}{n-1})$. This interval is non-empty for all $n_1 > 1$.

Equipped with findings from Proposition 3, we are now ready to compare between Bertrand and Cournot for consumer surplus, profits, and welfare. In a standard differentiated duopoly setting, Cournot competition yields higher profits, lower consumer surplus and lower welfare compared to Bertrand competition. Proposition 4 says that the results are *completely opposite* for consumer surplus and profits. Also note that the reversals (for consumer surplus as well as profits) hold irrespective of the extent of differentiation. However, as in the standard setting, welfare is lower in Cournot.

[5]See the finding (a) in Remark 2 though, which implies that irrespective of the number of public firms, a private firm's price is the same in Cournot and Bertrand if there is only one private firm.

[6]See Proposition 6.1 in Vives [21] for a comparison of elasticities under Bertrand and Cournot.

[7]For any $x \in \Re$,

$$\text{sgn}[x] = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Proposition 4. Suppose $U(q)$ is given by (5). Then for all $b \in (0,1)$ we have

(i) $\pi_i^{\mathcal{C}} < \pi_i^{\mathcal{B}}$, $i = \{1, 2\}$,
(ii) $CS^{\mathcal{C}} > CS^{\mathcal{B}}$,
(iii) $W^{\mathcal{C}} < W^{\mathcal{B}}$.

Proof: (i) Since $p_1^{\mathcal{C}} = m$, $\pi_1^{\mathcal{C}} = (p_1^{\mathcal{C}} - m)q_1^{\mathcal{C}} = 0$. Also, since $p_1^{\mathcal{B}} > m$ and $q_1^{\mathcal{B}} > 0$, $\pi_1^{\mathcal{B}} = (p_1^{\mathcal{B}} - m)q_1^{\mathcal{B}} > 0$. Thus $\pi_1^{\mathcal{C}} = 0 < \pi_1^{\mathcal{B}}$. From Proposition 3, we have (a) $p_2^{\mathcal{C}} - m = p_2^{\mathcal{B}} - m > 0$ and (b) $q_2^{\mathcal{C}} < q_2^{\mathcal{B}}$. Together, (a) and (b) imply $\pi_2^{\mathcal{C}} \equiv (p_2^{\mathcal{C}} - m)q_2^{\mathcal{C}} < (p_2^{\mathcal{B}} - m)q_2^{\mathcal{B}} \equiv \pi_2^{\mathcal{B}}$.
(ii) Since $p_2^{\mathcal{C}} = p_2^{\mathcal{B}}$, we have $CS^{\mathcal{B}} - CS^{\mathcal{C}} = -\int_{p_1^{\mathcal{C}}}^{p_1^{\mathcal{B}}} D_1(p_1, p_2^{\mathcal{B}})dp_1$. By Proposition 3 (i), $p_1^{\mathcal{B}} > p_1^{\mathcal{C}}$. Also, $D_1(p_1, p_2^{\mathcal{B}}) > 0$ for all $p_1 \in [p_1^{\mathcal{C}}, p_1^{\mathcal{B}}]$ since $D_1(p_1^{\mathcal{B}}, p_2^{\mathcal{B}}) = q_1^{\mathcal{B}} > 0$ and $\frac{\partial D_1(p_1, p_2^{\mathcal{B}})}{\partial p_1} < 0$. Thus $\int_{p_1^{\mathcal{C}}}^{p_1^{\mathcal{B}}} D_1(p_1, p_2^{\mathcal{B}})dp_1 > 0$ and hence, $CS^{\mathcal{B}} - CS^{\mathcal{C}} < 0$.
(iii) By Proposition 3 (i), $p_2^{\mathcal{C}} = p_2^{\mathcal{B}}$. Then the claim follows from applying Proposition 2. □

That profits could be lower under Cournot competition has also been shown in Häckner [11]. Crucial to Häckner's findings are the following features: the presence of strictly more than two firms and cost/quality asymmetry. None of these features are present in our framework. In our framework it is the presence of public firms that leads to the reversal of the profit and consumer surplus orderings (between Cournot and Bertrand competition).[8] These reversals also hold when the number of firms exceeds two for a range of $b \in (0,1)$. The fact that welfare reversal does not occur under linear demand structure is surprising, since Cournot seems to be more competitive according to several other indicators of competition.[9] For example, all prices are lower in a Cournot duopoly (Proposition 3 (i)). Also, the aggregate output is higher under Cournot, i.e., $q_1^{\mathcal{C}} + q_2^{\mathcal{C}} > q_1^{\mathcal{B}} + q_2^{\mathcal{B}}$.[10]

5. Bertrand versus Cournot in the Presence of Partial Privatization

In our comparison between Bertrand and Cournot outcomes, we have so far assumed that firm 1 (i.e., the public firm) maximizes welfare. The assumption is not strictly necessary to obtain reversals. Our results go through when firm 1 is partially privatized. To capture partial privatization, we

[8]See López and Naylor [12] for reversal of profit ordering in a unionized oligopoly setting.

[9]Although there are parameterizations such that all prices, public as well as private, are lower under Cournot, welfare reversal does not occur under any of those parameterizations.

[10]Concerning welfare ordering in Proposition 3 we find that if $U(q)$ is given by (5), $W(q) = s(q_1 + q_2) + d(q_1 - q_2)$, where $s(q_1 + q_2) = (a - m)(q_1 + q_2) - \frac{1}{2}(q_1 + q_2)^2$ and $d(q_1 - q_2) = -\frac{(1-b)}{4}(q_1 - q_2)^2$. Though $s'(.) > 0$, $d'(.) < 0$. That is, while an increase in aggregate output increases welfare, an increase in output differences between the two firms decreases welfare (since both varieties enter symmetrically into the utility function). Compared to Bertrand, $q_1 + q_2$ is higher in Cournot but $q_1 - q_2$ is higher as well. It turns out that the latter effect dominates, preserving the standard welfare ordering.

modify firm 1's objective function as follows. Firm 1 maximizes $R_1(q;\theta) \equiv \theta\pi_1(q) + (1-\theta)W(q)$ under Cournot and $\tilde{R}_1(p,\theta) = \theta\tilde{\pi}_1(p) + (1-\theta)\tilde{W}(p)$ under Bertrand where $\theta \in [0,1]$ and $\pi_1(q)$,$W(q)$,$\tilde{\pi}_1(p)$ and $\tilde{W}(p)$ are as defined in the previous section.[11] For a welfare-maximizing public firm, $\theta = 0$. If $\theta = 1$, the public firm is fully privatized while if $\theta \in (0,1)$, the public firm is partially privatized. It is easy to show that there exists $\tilde{\theta} > 0$ such that Proposition 3 (i) and 3 (ii), that is, $\pi_i^{\mathcal{C}} < \pi_i^{\mathcal{B}}$ for $i = 1,2$ and $CS^{\mathcal{C}} > CS^{\mathcal{B}}$, holds for $\theta < \tilde{\theta}$.

While the discussion above suggests that the reversal of Bertrand-Cournot ordering can occur in presence of partially privatized public firm, a limitation is that the degree of privatization, captured by the parameter θ, is exogenous. Consequently, the comparison between Bertrand and Cournot implicitly assumes that the degree of privatization is the same under the two modes of competition. This is not satisfactory since the incentives for privatization are typically different for Bertrand and Cournot.

To endogenize the degree of privatization we now construct a stylized two-stage game, where a welfare-maximizing government chooses $\theta \in [0,1]$ in stage 1, after which firms 1 and 2 compete in the product market in stage 2.[12]

5.1. **The Cournot game.** We consider a two-stage game. In stage 1, the social planner chooses $\theta \in [0,1]$ to maximize welfare. Given a stage 1 choice of $\theta \in [0,1]$, in stage 2, firm 1 chooses q_1 to maximize $R_1(q_1,q_2;\theta)$ and firm 2 chooses q_2 to maximize $\pi_2(q_1,q_2)$.

For any given $\theta \in [0,1]$ let $q^{\mathcal{C}}(\theta) = (q_1^{\mathcal{C}}(\theta), q_2^{\mathcal{C}}(\theta))$ denote output vector in stage 2 Cournot equilibrium. Then the following first order conditions must hold:

$$\frac{\partial R_1(q^{\mathcal{C}}(\theta);\theta)}{\partial q_1} = (p_1^{\mathcal{C}}(\theta) - c) + \theta q_1^{\mathcal{C}}(\theta)\frac{\partial P_1(q^{\mathcal{C}}(\theta))}{\partial q_1} = 0, \tag{9}$$

$$\frac{\partial \pi_2(q^{\mathcal{C}}(\theta))}{\partial q_2} = (p_2^{\mathcal{C}}(\theta) - c) + q_2^{\mathcal{C}}(\theta)\frac{\partial P_2^{\mathcal{C}}(q^{\mathcal{C}}(\theta))}{\partial q_2} = 0, \tag{10}$$

where $p_i^{\mathcal{C}}(\theta) \equiv P_i(q^{\mathcal{C}}(\theta)), i = 1,2$. Lemma 1 records the effect of privatization on outputs for later reference.

Lemma 1. Suppose $q_i^{\mathcal{C}}(\theta) > 0, i = 1,2$. Then $\frac{\partial q_1^{\mathcal{C}}(\theta)}{\partial \theta} < 0$ and $\frac{\partial q_2^{\mathcal{C}}(\theta)}{\partial \theta} > 0$.

When $\theta = 0$, the public firm's price equals marginal cost and hence the public firm earns zero profits. As θ increases, that is, as the weight attached to profits increases, the public firm finds it optimal to cut back production which raises its price above marginal cost. Lemma 1 says that indeed starting from any $\theta = \theta_0$, the public firm's output declines as θ increases. Since

[11] Similar formulations exist in the mixed oligopoly literature. See, for example, Matsumura [13] and Fershtman [10].

[12] Using the parameter θ to capture the degree of privatization is simplistic. Nevertheless, this is in line with the mixed oligopoly literature, where also, the importance of the profit motive of the public firm is used to capture the degree of privatization.

outputs are strategic substitutes, the private firm's output increases with an increase in θ.

Now consider the stage 1 choice of θ by a welfare maximizing government. Define $W^{\mathcal{C}}(\theta) = W(q^{\mathcal{C}}(\theta))$. Using Lemma 1 it is straightforward to establish the following:

Proposition 5. Suppose $\theta^{\mathcal{C}}$ maximizes $W^{\mathcal{C}}(\theta)$. Then $\theta^{\mathcal{C}} > 0$.

Proof: Since $W(q)$ is continuous in q and $q^{\mathcal{C}}(\theta)$ is continuous in θ, $W^{\mathcal{C}}(\theta)$ is continuous in θ over the compact interval $\theta \in [0, 1]$. Therefore, there exists $\theta^{\mathcal{C}} \in [0, 1]$ such that $W^{\mathcal{C}}(\theta)$ attains its maximum at $\theta = \theta^{\mathcal{C}}$. Differentiating $W^{\mathcal{C}}(\theta)$ with respect to θ yields:

$$\frac{dW^{\mathcal{C}}(\theta)}{d\theta} = (p_1^{\mathcal{C}}(\theta) - m)\left(\frac{\partial q_1^{\mathcal{C}}(\theta)}{\partial\theta}\right) + (p_2^{\mathcal{C}}(\theta) - m)\left(\frac{\partial q_2^{\mathcal{C}}(\theta)}{\partial\theta}\right). \tag{11}$$

We have $p_1^{\mathcal{C}}(0) - m = 0$, $p_2^{\mathcal{C}}(0) - m > 0$ and by Lemma 1, $\frac{\partial q_2^{\mathcal{C}}(0)}{\partial\theta} > 0$. Then it follows from (11) that $\frac{dW^{\mathcal{C}}(0)}{d\theta} > 0$, which in turn implies $\theta^{\mathcal{C}} > 0$. □

The intuition for Proposition 5 is simple. Consider an infinitesimally small increase in θ from $\theta = 0$. As the weight on its own profits increases, the public firm lowers its output, q_1, while the rival firm raises its output, q_2. The welfare loss from a reduction in q_1 is second order since $p_1^{\mathcal{C}}(0) - m = 0$ while the welfare gain from an increase in q_2 is first order since $p_2^{\mathcal{C}}(0) - m > 0$. This implies that there always exists a certain degree of privatization, which improves welfare when the second stage game is Cournot.

Note that although government maximizes welfare, it instructs the public firm to maximize something different: a weighted sum of its profits and welfare. This idea is familiar from the strategic delegation literature where managers are given incentives (by owners) to maximize a weighted sum of profits and sales even though the owners only care about profits (see, for example, Vickers [19], Fershtman and Judd [9], and Sklivas [18]). By assigning a strictly positive weight to sales in managers' incentive contracts, the profit-maximizing owner credibly commits to a *higher* output which in turn raises profits. Similarly, by assigning a strictly positive weight to profits in the public firm's objective function, the welfare-maximizing government credibly commits to a *lower* output (to be produced by the public firm) which in our framework raises welfare by partially correcting the underproduction by the private firm.

In homogenous products Cournot duopoly, Fershtman [10] and Matsumura [13] also found that partial privatization can improve welfare. However, for partial privatization to be strictly welfare improving in their setting, either the public firm has to be relatively inefficient or marginal cost has to be strictly increasing in output. None of these are necessary in a differentiated products duopoly setup like ours. Product differentiation alone, no matter how small, is sufficient to generate welfare-improving partial privatization under Cournot competition.

5.2. **The Bertrand game.** In stage 1, the government chooses $\theta \in [0, 1]$ to maximize $\tilde{W}(p)$. In stage 2, firm 1 chooses p_1 to maximize $\tilde{R}_1(p_1, p_2; \theta) \equiv \theta\tilde{\pi}_1(p) + (1 - \theta)\tilde{W}(p)$ and firm 2 chooses p_2 to maximize $\tilde{\pi}_2(p_1, p_2)$.

For any given $\theta \in [0,1]$ let $p^{\mathcal{B}}(\theta) = (p_1^{\mathcal{B}}(\theta), p_2^{\mathcal{B}}(\theta))$ denote the price vector in a stage 2 Bertrand equilibrium. Then the following first order conditions must hold:

$$\frac{\partial \tilde{R}_1(p^{\mathcal{B}}(\theta);\theta)}{\partial p_1} = (p_1^{\mathcal{B}}(\theta) - m)\frac{\partial D_1(p^{\mathcal{B}}(\theta))}{\partial p_1} + \theta q_1^{\mathcal{B}}(\theta) + (1-\theta)(p_2^{\mathcal{B}}(\theta) - m)\frac{\partial D_2(p^{\mathcal{B}}(\theta))}{\partial p_1} = 0, \tag{12}$$

$$\frac{\partial \tilde{\pi}_2(p^{\mathcal{B}}(\theta))}{\partial p_2} = (p_2^{\mathcal{B}}(\theta) - m)\frac{\partial D_2(p^{\mathcal{B}}(\theta))}{\partial p_2} + q_2^{\mathcal{B}}(\theta) = 0, \tag{13}$$

where $q_i^{\mathcal{B}}(\theta) \equiv D_i(p^{\mathcal{B}}(\theta))$, $i = 1, 2$.

Consider an infinitesimally small increase in θ from $\theta = 0$. Recall that in the Cournot game, introduction of this profit motive induced the public firm to reduce output. Similarly, here, suppose the profit motive induces the public firm to raise its price, p_1. Then private firm's price, p_2, increases as well since prices are strategic complements (Assumption 3). Given $p_i^{\mathcal{B}}(0) > m$ for both $i = 1, 2$, a further increase in prices triggered by partial privatization reduces welfare in Bertrand competition. While this conclusion seems natural and holds under linear demand, note that we started with the supposition that the introduction of the profit motive induces the public firm to raise its price, p_1, above $p_1^{\mathcal{B}}(0)$. Assumptions 1 - 3 do not guarantee that. A sufficient condition for the supposition to hold is that the public firm produces more than the private firm in absence of privatization. Lemma 2 and Proposition 6 summarize our discussion.

Lemma 2. Suppose $q_i^{\mathcal{B}}(0) > 0$ and furthermore $q_1^{\mathcal{B}}(0) > q_2^{\mathcal{B}}(0)$. Then $\frac{\partial p_i^{\mathcal{B}}(0)}{\partial \theta} > 0$, $i = 1, 2$.

Proof: See Appendix.

Proposition 6. Define $W^{\mathcal{B}}(\theta) = \tilde{W}(p^{\mathcal{B}}(\theta)) = W(q^{\mathcal{B}}(\theta))$. Now, suppose $q_i^{\mathcal{B}}(0) > 0$. Then $\frac{dW^{\mathcal{B}}(0)}{d\theta} < 0$ if $q_1^{\mathcal{B}}(0) > q_2^{\mathcal{B}}(0)$.

Proof: Differentiating $\tilde{W}(p^{\mathcal{B}}(\theta))$ with respect to θ and evaluating at $\theta = 0$ gives

$$\frac{d\tilde{W}(p^{\mathcal{B}}(0))}{d\theta} = \frac{\partial \tilde{W}(p^{\mathcal{B}}(0))}{\partial p_1}\frac{\partial p_1^{\mathcal{B}}(0)}{\partial \theta} + \frac{\partial \tilde{W}(p^{\mathcal{B}}(0))}{\partial p_2}\frac{\partial p_2^{\mathcal{B}}(0)}{\partial \theta}, \tag{14}$$

From first order conditions we get $\frac{\partial \tilde{W}(p^{\mathcal{B}}(0))}{\partial p_1} = 0$. Also, if $q_1^{\mathcal{B}}(0) > q_2^{\mathcal{B}}(0)$, $\frac{\partial p_2^{\mathcal{B}}(0)}{\partial \theta} > 0$ (Lemma 2). Hence it suffices to show that $\frac{\partial \tilde{W}(p^{\mathcal{B}}(0))}{\partial p_2} < 0$. We have

$$\frac{\partial \tilde{W}(p^{\mathcal{B}}(0))}{\partial p_2} = (p_1^{\mathcal{B}}(0) - m)\frac{\partial D_1(p^{\mathcal{B}}(0))}{\partial p_2} + (p_2^{\mathcal{B}}(0) - m)\frac{\partial D_2(p^{\mathcal{B}}(0))}{\partial p_2}. \tag{15}$$

From $\frac{\partial \tilde{W}(p^{\mathcal{B}}(0))}{\partial p_1} = (p_1^{\mathcal{B}}(0) - m)\frac{\partial D_1(p^{\mathcal{B}}(0))}{\partial p_1} + (p_2^{\mathcal{B}}(0) - m)\frac{\partial D_2(p^{\mathcal{B}}(0))}{\partial p_1} = 0$, we get $(p_1^{\mathcal{B}}(0) - m) = -\frac{(p_2^{\mathcal{B}}(0)-m)\frac{\partial D_2(p^{\mathcal{B}}(0))}{\partial p_1}}{\frac{\partial D_1(p^{\mathcal{B}}(0))}{\partial p_i}}$. Substituting this in (15), using (1) and then simplifying further gives $\frac{\partial \tilde{W}(p^{\mathcal{B}}(0))}{\partial p_2} = \frac{p_2^{\mathcal{B}}(0)-m}{U_{22}(q^{\mathcal{B}}(0))}$. Since $p_2^{\mathcal{B}}(0) - m > 0$ and $U_{22} < 0$, it follows that $\frac{\partial \tilde{W}(p^{\mathcal{B}}(0))}{\partial p_2} < 0$. $\square$

Proposition 6 says that small increments in θ from $\theta = 0$ reduce welfare under Bertrand competition. What about large increments? For a linear demand system we find that they are not welfare improving either. Using CES preferences, Anderson et al. [2] have shown that full privatization, that is, a change from $\theta = 0$ to $\theta = 1$, reduces welfare in Bertrand competition. Using the standard quadratic utility specification, Proposition 7 shows that not only full privatization, but no extent of privatization can improve welfare in Bertrand competition.

5.3. **Linear demand:** Before stating our finding more formally, let us define $\theta^{\mathcal{B}}$ to be the value of $\theta \in [0,1]$ that maximizes $\tilde{W}^{\mathcal{B}}(\theta)$. Recall $\theta^{\mathcal{C}}$ denotes the optimal degree of privatization under Cournot. Proposition 7 below compares $\theta^{\mathcal{B}}$ and $\theta^{\mathcal{C}}$ for linear demand.

Proposition 7. Suppose $U(q)$ is given by (5). Then $\theta^{\mathcal{C}} \in (0,1)$ while $\theta^{\mathcal{B}} = 0$.

Proof: See Appendix.

Finally we turn to the comparison between Bertrand and Cournot outcomes in this two-stage game with an endogenous degree of privatization. Define $\pi_i^{\mathcal{C}}(\theta) \equiv \pi_i(q^{\mathcal{C}}(\theta))$,$\pi_i^{\mathcal{B}}(\theta) \equiv \tilde{\pi}_i(p^{\mathcal{B}}(\theta)) \equiv \pi_i(q^{\mathcal{B}}(\theta)), CS^{\mathcal{C}}(\theta) \equiv CS(q^{\mathcal{C}}(\theta))$ and $CS^{\mathcal{B}}(\theta) \equiv CS(q^{\mathcal{B}}(\theta))$. As in Proposition 4, we find a reversal of standard Bertrand-Cournot ordering for consumer surplus and profits under a range of parameterizations. Welfare, as in section 4, remains higher under Bertrand.

Proposition 8. Suppose $U(q)$ is given by (5). Then

(i) $\theta^{\mathcal{C}} = \frac{b(1-b)}{(4-3b)} > 0, \theta^{\mathcal{B}} = 0$;
(ii) $\pi_2^{\mathcal{C}}(\theta^{\mathcal{C}}) < \pi_2^{\mathcal{B}}(\theta^{\mathcal{B}})$;
(iii) $\pi_1^{\mathcal{C}}(\theta^{\mathcal{C}}) < \pi_1^{\mathcal{B}}(\theta^{\mathcal{B}})$ and $CS^{\mathcal{C}}(\theta^{\mathcal{C}}) > CS^{\mathcal{B}}(\theta^{\mathcal{B}})$ if $b \in (0, 0.84)$, and finally
(iv) $W^{\mathcal{C}}(\theta^{\mathcal{C}}) < W^{\mathcal{B}}(\theta^{\mathcal{B}})$.

Proof: See Appendix.

6. Summary and Concluding Remarks

Comparing Bertrand and Cournot equilibrium outcomes in a differentiated duopoly with a welfare-maximizing public firm, we have shown that the standard Bertrand-Cournot rankings could be reversed for prices, consumer surplus, and profits. The reversals also hold under a richer setting with a partially privatized public firm, where the extent of privatization is endogenously determined by a welfare-maximizing government. Partial privatization can have different welfare implications for Bertrand and Cournot competition. In particular, partial privatization (to a certain extent) always improves welfare under Cournot competition but not necessarily so under Bertrand competition.

Note that some of our results are obtained using linear demand structure and constant marginal cost. Clearly, whether these results hold for other demand structures or other cost specifications requires further investigation. However, if the public firm maximizes welfare, the possibility of reversals remain since the underlying reason (for these reversals) — public firm's lower price under Cournot — holds irrespective of the demand or cost specification.

Throughout the analysis we have assumed that the two goods are imperfect substitutes as this is the predominant case considered in the literature. Instead, if the two goods are complements, then reversals are unlikely since the public firm's price is lower (at least weakly) in Bertrand than in Cournot competition. As in Proposition 1, the public firm's price equals marginal cost under Cournot competition. On the other hand, under Bertrand, the public firm's price is either equal or strictly less than marginal cost. To see this, consider an infinitesimally small decline in the public firm's price from marginal cost. This increases the public firm's output and since two goods are complements, the private firm's output increases as well. As there is an underproduction of the private good, the increase in the private firm's output generates a first order welfare gain. Therefore, a welfare-maximizing public firm will set its price strictly lower than marginal cost if it is allowed to make losses, and equal to marginal cost otherwise.

7. Appendix

Proof of Lemma 1: Totally differentiating (9) and (10) with respect to θ and then solving for $\frac{\partial q_1^{\mathcal{C}}(\theta)}{\partial \theta}$ and $\frac{\partial q_2^{\mathcal{C}}(\theta)}{\partial \theta}$ we get

$$\frac{\partial q_1^{\mathcal{C}}(\theta)}{\partial \theta} = \frac{-q_1^{\mathcal{C}}(\theta)\left(\frac{\partial P_1(q^{\mathcal{C}}(\theta))}{\partial q_1}\right)\left(\frac{\partial^2 \pi_2(q^{\mathcal{C}}(\theta))}{\partial q_2^2}\right)}{\Delta}, \tag{16}$$

$$\frac{\partial q_2^{\mathcal{C}}(\theta)}{\partial \theta} = \frac{q_1^{\mathcal{C}}(\theta)\left(\frac{\partial P_1(q^{\mathcal{C}}(\theta))}{\partial q_1}\right)\left(\frac{\partial^2 \pi_2(q^{\mathcal{C}}(\theta))}{\partial q_2 \partial q_1}\right)}{\Delta}. \tag{17}$$

The following second order conditions must hold: $\frac{\partial^2 \pi_2(q^{\mathcal{C}}(\theta))}{\partial q_2^2} < 0$, and $\Delta = \left(\frac{\partial^2 R_1(q^{\mathcal{C}}(\theta);\theta)}{\partial q_1^2}\right)\left(\frac{\partial^2 \pi_2(q^{\mathcal{C}}(\theta))}{\partial q_2^2}\right) - \left(\frac{\partial^2 R_1(q^{\mathcal{C}}(\theta);\theta)}{\partial q_2 \partial q_1}\right)\left(\frac{\partial^2 \pi_2(q^{\mathcal{C}}(\theta))}{\partial q_2 \partial q_1}\right) > 0$. Then the result follows from noting that $q_1^{\mathcal{C}}(\theta) > 0$, $\frac{\partial P_1(q^{\mathcal{C}}(\theta))}{\partial q_1} < 0$ and $\frac{\partial^2 \pi_2(q^{\mathcal{C}}(\theta))}{\partial q_1 \partial q_2} < 0$ (Assumption 2 (ii)). $\square$

Proof of Lemma 2: Totally differentiating conditions (12) and (13) with respect to θ and then solving for $\frac{\partial p_i^{\mathcal{B}}(\theta)}{\partial \theta}$, we get

$$\frac{\partial p_1^{\mathcal{B}}(\theta)}{\partial \theta} = \frac{\left[(p_2^{\mathcal{B}}(\theta) - m)\frac{\partial D_2(p^{\mathcal{B}}(\theta))}{\partial p_1} - q_1^{\mathcal{B}}(\theta)\right]\left(\frac{\partial^2 \tilde{\pi}_2(p^{\mathcal{B}}(\theta))}{\partial p_2^2}\right)}{\tilde{\Delta}} \tag{18}$$

$$\frac{\partial p_2^{\mathcal{B}}(\theta)}{\partial \theta} = \frac{-\left[(p_2^{\mathcal{B}}(\theta) - m)\frac{\partial D_2(p^{\mathcal{B}}(\theta))}{\partial p_1} - q_1^{\mathcal{B}}(\theta)\right]\left(\frac{\partial^2 \tilde{\pi}_2(p^{\mathcal{B}}(\theta))}{\partial p_2 \partial p_1}\right)}{\tilde{\Delta}} \tag{19}$$

where $\tilde{\Delta} = \left(\frac{\partial^2 \tilde{R}_1(p^{\mathcal{B}}(\theta);\theta)}{\partial p_1^2}\right)\left(\frac{\partial^2 \tilde{\pi}_2(p^{\mathcal{B}}(\theta))}{\partial p_2^2}\right) - \left(\frac{\partial^2 \tilde{R}_1(p^{\mathcal{B}}(\theta);\theta)}{\partial p_2 \partial p_1}\right)\left(\frac{\partial^2 \tilde{\pi}_2(p^{\mathcal{B}}(\theta))}{\partial p_2 \partial p_1}\right)$. Substituting $p_2^{\mathcal{B}}(\theta) - m = \frac{q_2^{\mathcal{B}}(\theta)}{\left(-\frac{\partial D_2^{\mathcal{B}}(\theta)}{\partial p_2}\right)}$ (from (13)) and $\frac{\partial D_i(p^{\mathcal{B}}(\theta))}{\partial p_j} = \frac{U_{ij}(q^{\mathcal{B}}(\theta))}{U_{11}(q^{\mathcal{B}}(\theta))U_{22}(q^{\mathcal{B}}(\theta)) - U_{12}(q^{\mathcal{B}}(\theta))U_{21}(q^{\mathcal{B}}(\theta))}$ in (19) and then evaluating at $\theta = 0$ we get

$$\frac{\partial p_1^{\mathcal{B}}(0)}{\partial \theta} = \frac{q_2^{\mathcal{B}}(0)\left(\frac{q_1^{\mathcal{B}}(0)}{q_2^{\mathcal{B}}(0)} - \frac{U_{12}(q^{\mathcal{B}}(0))}{U_{22}(q^{\mathcal{B}}(0))}\right)\left(-\frac{\partial^2 \tilde{\pi}_2(p^{\mathcal{B}}(0))}{\partial p_2^2}\right)}{\tilde{\Delta}}, \tag{20}$$

$$\frac{\partial p_2^{\mathcal{B}}(0)}{\partial \theta} = \frac{q_2^{\mathcal{B}}(0)\left(\frac{q_1^{\mathcal{B}}(0)}{q_2^{\mathcal{B}}(0)} - \frac{U_{12}(q^{\mathcal{B}}(0))}{U_{22}(q^{\mathcal{B}}(0))}\right)\left(\frac{\partial^2 \tilde{\pi}_2(p^{\mathcal{B}}(0))}{\partial p_2 \partial p_1}\right)}{\tilde{\Delta}}. \tag{21}$$

Note $q_2^{\mathcal{B}}(0) > 0$. By Assumption 3 (ii), $\frac{\partial^2 \tilde{\pi}_2(p^{\mathcal{B}}(0))}{\partial p_2 \partial p_1} > 0$. By Assumption 1 (v), $\frac{U_{12}(q^{\mathcal{B}}(0))}{U_{22}(q^{\mathcal{B}}(0))} < 1$. Since second-order conditions are satisfied at $p = p^{\mathcal{B}}(\theta)$, $\tilde{\Delta} > 0$ and $-\frac{\partial^2 \tilde{\pi}_2(p^{\mathcal{B}}(0))}{\partial p_2^2} > 0$. Now, if $\frac{q_1^{\mathcal{B}}(0)}{q_2^{\mathcal{B}}(0)} > 1$ we have $\frac{q_1^{\mathcal{B}}(0)}{q_2^{\mathcal{B}}(0)} - \frac{U_{12}(q^{\mathcal{B}}(0))}{U_{22}(q^{\mathcal{B}}(0))} > 0$. Thus all expressions in the right-hand side of (20) and (21) are strictly positive which in turn implies $\frac{\partial p_i^{\mathcal{B}}(0)}{\partial \theta} > 0$, $i = 1, 2$. □

Proof of Proposition 7: Routine calculations show that, if $U(q)$ is given by (5), then for any given $\theta \in [0, 1]$, stage 2 Cournot equilibrium quantities are:

$$q_1^{\mathcal{C}}(\theta) = \frac{(2-b)(a-m)}{(2+2\theta-b^2)}, \quad q_2^{\mathcal{C}}(\theta) = \frac{(1+\theta-b)(a-m)}{(2+2\theta-b^2)}.$$

Using $W^{\mathcal{C}}(\theta) = W(q^{\mathcal{C}}(\theta)) = U(q_1^{\mathcal{C}}(\theta), q_2^{\mathcal{C}}(\theta)) - m(q_1^{\mathcal{C}}(\theta) + q_2^{\mathcal{C}}(\theta))$ we get

$$W^{\mathcal{C}}(\theta) = \frac{[(7-6b-2b^2+2b^3) + \theta(14-10b) + 3\theta^2](a-m)^2}{2(2+2\theta-b^2)^2}. \tag{22}$$

Existence of $\theta^{\mathcal{C}}$ follows from continuity of $W^{\mathcal{C}}(\theta)$ in θ over the compact interval $[0, 1]$. Proposition 5 gives $\theta^{\mathcal{C}} > 0$. Since $\frac{dW^{\mathcal{C}}(1)}{d\theta} = -\frac{(a-m)^2}{(2+b)^3} < 0$, $\theta^{\mathcal{C}} < 1$. Thus $\theta^{\mathcal{C}} \in (0, 1)$.

The Bertrand equilibrium prices, given any stage 1 choice of θ, are:

$$p_1^{\mathcal{B}}(\theta) = m + \frac{(1-b)(2\theta+b)(a-m)}{(2+2\theta-b^2)}, \quad p_2^{\mathcal{B}}(\theta) = m + \frac{(1-b)(1+\theta+b\theta)(a-m)}{(2+2\theta-b^2)}.$$

Then using $q_i^{\mathcal{B}}(\theta) = D_i(p^{\mathcal{B}}(\theta))$ and $W^{\mathcal{B}}(\theta) \equiv W^{\mathcal{B}}(q^{\mathcal{B}}(\theta)) = U(q_1^{\mathcal{B}}(\theta), q_2^{\mathcal{B}}(\theta)) - m(q_1^{\mathcal{B}}(\theta) + q_2^{\mathcal{B}}(\theta))$ we get

$$q_1^{\mathcal{B}}(\theta) = \frac{(2+b\theta-b^2(1-\theta)(a-m))}{(1+b)(2+2\theta-b^2)}, \quad q_2^{\mathcal{B}}(\theta) = \frac{(1+\theta+b\theta)(a-m)}{(1+b)(2+2\theta-b^2)},$$

and

$$W^{\mathcal{B}}(\theta) = \frac{f(\theta)(a-m)^2}{2(1+b)(2+2\theta-b^2)^2}, \tag{23}$$

where $f(\theta) = 7 + b - 7b^2 - b^3 + 2b^4 + \theta(14 - 4b^2 - 2b^4) + \theta^2(3 + 7b + b^2 - 3b^3)$. Existence of $\theta^{\mathcal{B}}$ follows from continuity of $W^{\mathcal{B}}(\theta)$ in θ over the compact interval [0,1]. Observe that $q_i^{\mathcal{B}}(0) > 0$, for $i = 1, 2$ and $q_1^{\mathcal{B}}(0) = \frac{a-m}{1+b} > \frac{a-m}{(1+b)(2-b^2)} = q_2^{\mathcal{B}}(0)$. Then applying Proposition 6 we get $\frac{dW^{\mathcal{B}}(0)}{d\theta} < 0$. Indeed $\frac{dW^{\mathcal{B}}(\theta)}{d\theta} = -\frac{(2+b)(1-b)^2(b(1+b)+\theta(4+3b))(a-m)^2}{(2+2\theta-b^2)^3} < 0$ for all $\theta \in [0, 1]$, which implies $\theta^{\mathcal{B}} = 0$. □

Proof of Proposition 8: i) Differentiating (22) gives $\frac{dW^{\mathcal{C}}(\theta)}{d\theta} = -\frac{(2-b)(b(1-b)-\theta(4-3b))(a-m)^2}{(2+2\theta-b^2)^3}$ which equals zero at $\theta = \frac{b(1-b)}{(4-3b)}$. Since $\frac{d^2W^{\mathcal{C}}(\theta)}{d\theta^2} = -\frac{(2-b)(4+4b+3b^3-10b^2)(a-m)^2}{(2+2\theta-b^2)^4} < 0$ at $\theta = \frac{b(1-b)}{(4-3b)}$, and $\theta^{\mathcal{C}} \in (0, 1)$ by Proposition 5, it follows that $W^{\mathcal{C}}(\theta)$ attains its maximum at $\theta = \frac{b(1-b)}{(4-3b)}$. Thus $\theta^{\mathcal{C}} = \frac{b(1-b)}{(4-3b)}$. By Proposition 7, $\theta^{\mathcal{B}} = 0$.

(ii) Using the values of $q_i^{\mathcal{C}}(\theta)$, $i = 1, 2$ from the proof of Proposition 6 we find that $\pi_1^{\mathcal{C}}(\theta^{\mathcal{C}}) = \frac{b(1-b)(4-3b)(a-m)^2}{(4-3b^2)^2}$, $\pi_2^{\mathcal{C}}(\theta^{\mathcal{C}}) = \frac{4(1-b)^2(a-m)^2}{(4-3b^2)^2}$, $CS^{\mathcal{C}}(\theta^{\mathcal{C}}) = \frac{(5-4b)(a-m)^2}{2(4-3b^2)}$ and $W^{\mathcal{C}}(\theta^{\mathcal{C}}) = \frac{(7-6b)(a-m)^2}{2(4-3b^2)}$.

Since $\theta^{\mathcal{B}} = 0$, $p_i^{\mathcal{B}}(\theta^{\mathcal{B}})$ and $q_i^{\mathcal{B}}(\theta^{\mathcal{B}})$, $i = 1, 2$ are same as in the proof of Lemma 1. Using those values we get $\pi_1^{\mathcal{B}}(\theta^{\mathcal{B}}) = \frac{b(1-b)(a-m)^2}{(1+b)(2-b^2)}$, $\pi_2^{\mathcal{B}}(\theta^{\mathcal{B}}) = \frac{(1-b)(a-m)^2}{(1+b)(2-b^2)^2}$, $CS^{\mathcal{B}}(\theta^{\mathcal{B}}) = \frac{(5-b-3b^2+b^3)(a-m)^2}{2(1+b)(2-b^2)^2}$ and $W^{\mathcal{B}}(\theta^{\mathcal{B}}) = \frac{(7+b-7b^2-b^3+2b^4)(a-m)^2}{2(1+b)(2-b^2)^2}$.

Now we turn to the comparison between Bertrand and Cournot outcomes. Observe that $\pi_2^{\mathcal{C}}(\theta^{\mathcal{C}}) - \pi_2^{\mathcal{B}}(\theta^{\mathcal{B}}) = -\frac{(1-b)b^2[1+(1-b^2)(7-4b^2)](a-m)^2}{(1+b)(2-b^2)^2(4-3b^2)^2} < 0$ for all $b \in (0, 1)$. Moreover, $\pi_1^{\mathcal{C}}(\theta^{\mathcal{C}}) - \pi_1^{\mathcal{B}}(\theta^{\mathcal{B}}) = -\frac{b(1-b)[8-2b-14b^2+b^3+6b^4](a-m)^2}{(1+b)(2-b^2)(4-3b^2)^2}$ which is strictly negative for $b \in (0, 0.84)$. We have $CS^{\mathcal{C}}(\theta^{\mathcal{C}}) - CS^{\mathcal{B}}(\theta^{\mathcal{B}}) = \frac{b(1-b)(8-b-12b^2+4b^4)(a-m)^2}{2(1+b)(2-b^2)^2(4-3b^2)}$, which is strictly positive for $b \in (0, 0.9)$. Thus for all $b \in (0, 0.84)$ we have $\pi_1^{\mathcal{C}}(\theta^{\mathcal{C}}) < \pi_1^{\mathcal{B}}(\theta^{\mathcal{B}})$ and $CS^{\mathcal{C}}(\theta^{\mathcal{C}}) > CS^{\mathcal{B}}(\theta^{\mathcal{B}})$.

Finally, the welfare comparison between Bertrand and Cournot shows that $W^{\mathcal{C}}(\theta^{\mathcal{C}}) - W^{\mathcal{B}}(\theta^{\mathcal{B}}) = -\frac{b^2(1-b)(3-2b^2)(a-m)^2}{2(1+b)(2-b^2)^2(4-3b^2)} < 0$. □

References

[1] Amir, R., Jin, J. Y., 2001. Cournot and Bertrand equilibria compared: substitutability, complementarity and concavity. International Journal of Industrial Organization 19, 303-317.

[2] Anderson, S. P., de Palma, A., Thisse, J.-F., 1997. Privatization and efficiency in a differentiated industry. European Economic Review 41, 1635-1654.

[3] Basu,K., 1993. Lectures in Industrial Organization Theory. Blackwell, Oxford UK & Cambridge USA.

[4] Cheng, L., 1985. Comparing Bertrand and Cournot equilibria: a geometric approach. Rand Journal of Economics 16, 146-152.

[5] Dastidar, K. G., 1997. Comparing Cournot and Bertrand in a homogeneous product market. Journal of Economic Theory 75, 205-212.

[6] De Fraja, G., Delbono, F., 1989. Alternative strategies of a public enterprise in oligopoly. Oxford Economic Papers 41, 302-311.

[7] De Fraja, G., Delbono, F., 1990. Game-theoretic models of mixed oligopoly. Journal of Economic Surveys 4, 1-17.

[8] Dixit, A., 1979. A model of duopoly suggesting a theory of entry barriers. The Bell Journal of Economics 10, 20-32.

[9] Fershtman, C., Judd, K., 1984. Equilibrium incentives in oligopoly. American Economic Review 77, 927-940.

[10] Fershtman, C., 1990. The interdependence between ownsership status and market structure: the case of privatization. Economica 57, 319-328.

[11] Häckner, J., 2000. A note on price and quantity competition in differentiated oligopolies. Journal of Economic Theory 93, 233-239.

[12] López, M.C., Naylor R. A., 2004. The Cournot-Bertrand profit differential: A reversal result in a differentiated duopoly with wage bargaining. European Economic Review 48, 681-696.

[13] Matsumura, T., 1998. Partial privatization in mixed duopoly. Journal of Public Economics 70, 473-483.

[14] Okuguchi, K., 1987. Equilibrium Prices in the Bertrand and Cournot Oligopolies. Journal of Economic Theory 42, 128-139.

[15] Qiu, L., 1997. On the Dynamic Efficiency of Bertrand and Cournot Equilibria. Journal of Economic Theory 75, 213-229.

[16] Schmitz Jr., J. A., 2001. Government production of investment goods and aggregate labor productivity. Journal of Monetary Economics 47, 163-187.

[17] Singh, N., Vives, X., 1984. Price and quantity competition in a differentiated duopoly. Rand Journal of Economics 15, 546-554.

[18] Sklivas, S. D., 1987. The strategic choice of managerial incentives. Rand Journal of Economics 18, 452-458.

[19] Vickers, J., 1985. Delegation and the Theory of the Firm. Economic Journal, Supplement 95, 138-147.

[20] Vives, X., 1985. On the efficiency of Bertrand and Cournot equilibria with product differentiation. Journal of Economic Theory 36, 166-175.

[21] Vives, X., 1999. Oligopoly Pricing: Old Ideas and New Tools. MIT Press, Cambridge and London.

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