Preferred Suppliers and Vertical Integration in Auction Markets

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Abstract

This paper examines a symmetric first-price procurement auction in which one supplier is preferred by the buyer. Preference takes the form of a *right-of-first-refusal* which allows the preferred supplier to accept or reject the contract at the lowest bid of the other competing suppliers. We first characterize a strictly monotonic bidding function for the competing suppliers. We then show that the buyer can benefit from selling preference to one of the suppliers. The sale of preference allows the buyer to extract the differential expected profits from becoming the preferred supplier. Preference can also arise from a vertical merger between the buyer and one of the suppliers. We also show that the joint surplus of the buyer and one supplier increases after the vertical merger.

1.   Introduction

In this paper, we investigate the incentives for vertical restraints in a market characterized by procurement auctions. Corporate buyers frequently have a preference for particular suppliers of various inputs. Preference may arise from a variety of vertical arrangements. Preference for one supplier may arise from contractual relationships between the buyer and a particular supplier. Alternatively, preference may be the natural consequence of vertical integration between the buyer and one supplier.

Preference occurs when the buyer creates a *right-of-first-refusal* for one supplier which allows this preferred supplier to accept the contract at a price equal to the lowest bid by the other competing suppliers. The preferred supplier will clearly accept the contract whenever his cost is below the lowest bid of the competing suppliers. Bikhchandani, Lippman, and Ryan (2005) have recently examined a *right-of-first-refusal* in a symmetric sealed-bid second-price auction. In a private-value setting, they find that the gains of a buyer receiving a *right-of-first-refusal* are exactly equivalent to the loss of the seller of the good awarding a *right-of-first-refusal*. Thus, they conclude that there is no incentive for the seller to award a *right-of-first-refusal* to one of the buyers. An immediate corollary is that there would be no incentive for vertical integration between the seller and one of the buyers.

As we show, this conclusion depends on the particular auction mechanism, that is, a second-price auction. Without preference, symmetric first-price and second-price auctions are equivalent in the setting of independent private values. However, with preference for one supplier, first-price and second-price auctions are not equivalent in the allocation of the contracts.[1] Irrespective of the source of the preference, either a *right-of-first-refusal* or vertical integration, the other competing suppliers face a random reserve price equal to the cost of the preferred supplier. This random reserve price alters the allocation of the contract between the preferred supplier and the competing suppliers differently for a first-price auction than for a second-price auction. Therefore, the expected surplus or profits of the buyer and all of the suppliers are affected by the auction mechanism. As a result, the incentive of the buyer to award a *right-of-first-refusal* or to integrate vertically is altered.

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[1] Even second-price auctions and oral-ascending auctions would not be equivalent (see Brisset and Naegelen 2006).

In a symmetric first-price procurement auction with private values (costs), we show that when the number of suppliers is small there is always an incentive for the buyer and any one of the suppliers to integrate vertically. That is, the sum of the buyer's surplus and the expected profit of the preferred supplier exceeds that which would occur without preference. Similarly, if the buyer auctions a *right-of-first-refusal*, the symmetric suppliers would bid the price for preference upward to the point where they are each indifferent between winning and losing the auction for preference. The profit of the other competing suppliers is unambiguously lower as a result of preference awarded to one supplier. This then implies that the profit of the preferred supplier, net of the payment to the buyer for preference, is also lower with preference. Thus, if the buyer holds an auction for a *right-of-first-refusal*, then the buyer can appropriate more than the increase in the combined surplus that would arise from a vertical merger of the buyer and one supplier. With relatively mild conditions on the cost distribution of the suppliers, these results can be extended to any arbitrary number of potential suppliers. In particular, when the inverse hazard rate of the cost distribution is decreasing and convex, we show that the buyer's surplus from selling preference and the combined surplus of the buyer and preferred supplier are both higher with preference.

Although preference may benefit the buyer, preference creates a social cost because the contract is assigned to the preferred supplier more often than would be efficient. With a *right-of first-refusal*, the preferred supplier may obtain the contract even when some other supplier has a lower cost. This occurs when the cost of the preferred supplier is less than the lowest bid of the competing suppliers, but greater than the cost of the supplier with the lowest bid.[2]

The model posits a multi-stage game between the buyer and the suppliers. The decision to sell a *right-of-first-refusal* or to integrate vertically is made before the procurement needs of the buyer are known. Moreover, this decision is made under symmetric information about the common cost distribution of the suppliers, but before each supplier has realized its cost. Given the symmetry of the suppliers at this first stage, an optimal mechanism for the buyer would have the buyer commit to an efficient procurement auction without preference and charge each supplier an entry fee equal to its expected profits from that auction. However, optimal entry fees

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[2] This inefficiency is inescapable when preference arises as a consequence of vertical integration due. In that case, bargaining between the buyer and independent sellers would be characterized by two-sided asymmetric information. See Myerson and Satterthwaite (1983).

would require a degree of commitment and bargaining power on the part of the buyer that is difficult to envision and well beyond the commitment required for our preference auction. Indeed, optimal entry fees would require that all the suppliers which will participate in the procurement auction are present when the entry fees must be paid. Thus, there can be no subsequent entry or uncertainty about the number of future suppliers. Perhaps more importantly, optimal entry fees would require that the buyer, even before its procurement needs are known, commit to ignore future offers from any existing supplier that has refused to pay the entry fee. To the contrary, once its procurement needs are known, the buyer would have an incentive to entertain offers from as many suppliers as possible. Selling a *right-of-first-refusal* or dividing the surplus from a vertical merger does not require nearly this degree of commitment by the buyer or the suppliers. Every supplier values preference, but no competing supplier has to pay an entry fee for the right to submit a bid to the buyer. In addition, the buyer's incentive to sell preference or to integrate vertically with one supplier would not be eliminated by either entry of new suppliers before the procurement auction or by uncertainty over the number of suppliers which will participate in the procurement auction. Thus, the only requirement is that at least one supplier is present to receive preference at the time preference is awarded. Finally, the informational requirements imposed on the buyer by the creation of preference for one supplier are minimal. In particular, there is no need to calculate the optimal entry fee, which would be equal to the expected value of the difference between the first-order and second-order statistic on a number of draws from the cost distribution of the suppliers. If there are at least two suppliers present at the time preference is awarded, the buyer need only hold a pre-auction for preference prior to the procurement auction.

The rest of the paper proceeds as follows. In Section 2, we briefly survey the literature on exclusive dealing and vertical foreclosure that is most related to our paper. In Section 3, we construct a procurement auction with preference for one of several potential suppliers. The competing suppliers bid for the contract, while the preferred supplier has the right to accept or reject the contract at the lowest bid of the competing suppliers. For simplicity, we call this a preference auction. We then characterize the equilibrium bidding function for the competing suppliers, and prove that this bidding function is a strictly monotone equilibrium for the preference auction. In Section 4, we define the expected profits of the suppliers in the preference auction and compare them to the expected profits in an efficient first-price auction without

preference. We find that the expected profits of the preferred supplier exceed the expected profits in an efficient auction, which in turn exceed the expected profits of the competing suppliers in the preference auction. In Section 5, we identify sufficient conditions for the surplus of the buyer to be higher after selling preference. In particular, the payment for preference prior to the procurement auction can more than compensate the buyer for the higher expected price in the preference auction. In Section 6, we also show that under the same sufficient conditions, the combined surplus of the buyer and the preferred supplier will increase with preference. Thus, there are incentives for the buyer and one supplier to agree on preference or to merge vertically. Section 7 provides some concluding remarks.

2.   Related Literature on Exclusive Dealing and Vertical Integration

The model of preference by a buyer for one supplier in a procurement auction is analogous to exclusive dealing or vertical integration between an upstream manufacturer and a monopoly downstream distributor. A number of authors have examined models in which two manufacturers compete to foreclose each other by executing an exclusive dealing contract with a downstream distributor who has a monopoly in some final market. These papers employ an explicit or implicit model of consumer demand with differentiated goods, and focus on vertical foreclosure of the other manufacturer. This literature includes papers by Bernheim and Whinston (1986), Mathewson and Winter (1987)[3], Besanko and Perry (1994), Martimort (1996), O'Brien and Shaffer (1997), and Bernheim and Whinston (1998).[4] The two most recent of these papers examine a similar reduced-form model and obtain the same key result relevant for this paper.

In O'Brien and Shaffer (1997) and Bernheim and Whinston (1998), two manufacturers, denoted A and B, offer exclusive or non-exclusive contracts to a monopoly retailer. Let Π^C be

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[3]   See also the original paper by Comanor and Frech (1985) and another comment by Schwartz (1987).

[4]   Other papers on exclusive dealing have focused on issues of investment by buyers and sellers. This literature is primarily derived from the paper by Grossman and Hart (1986). See also Bolton and Whinston (1987) and more recently Segal and Whinston (2000). Our model does not include investment by the suppliers or the buyer. If investment was incorporated in our model after preference is sold but before the suppliers obtain their costs for the procurement auction, the preferred supplier is likely to have a stronger incentive to invest in technology that would generate a cost distribution which stochastically dominates the other competing suppliers. See Burguet and Perry (2000).

the maximized total surplus when the retailer is a common dealer for both manufacturers, and let Π^A and Π^B be the maximized total surplus when the retailer has an exclusive contract with one manufacturer, so that the other manufacturer is then foreclosed from this final market. The most interesting case arises when (1) $\Pi^C > \Pi^A > \Pi^B$, but (2) $\Pi^A + \Pi^B > \Pi^C$. The first inequalities imply that exclusivity reduces total surplus, and that the good of manufacturer A is more profitable alone than the good of manufacturer B. The second inequality implies that the goods are substitutes and not complements. Both assumptions would be satisfied naturally in a model of differentiated goods. Both papers demonstrate that there exists a Pareto undominated equilibrium for the manufacturers in which efficient common distribution occurs and each manufacturer receives a payment equal to its marginal contribution to the maximized total surplus. Manufacturer A receives $\Pi^C - \Pi^B$; manufacturer B receives $\Pi^C - \Pi^A$; and the retailer retains the remainder of the surplus equal to $\Pi^A + \Pi^B - \Pi^C$.

These papers then examine the cases in which one or both manufacturers offer only an exclusive contract and the retailer then chooses to be an exclusive dealer of one manufacturer, thereby foreclosing the other manufacturer completely. In the process of this discussion, both papers make the point that the retailer could increase his profits if he could hold an auction for his exclusive dealership. In this case, the retailer would receive a payment of Π^B to be the exclusive dealer of manufacturer A. This payment clearly exceeds the profit that the retailer would receive as a common distributor in the undominated equilibrium ($\Pi^B > \Pi^A + \Pi^B - \Pi^C$).[5] Thus, by auctioning an exclusive contract, the retailer could increase its profits by $\Pi^C - \Pi^A$ at the expense of manufacturer B, or equivalently at the expense of manufacturer A $[(\Pi^C - \Pi^B) - (\Pi^A - \Pi^B) = \Pi^C - \Pi^B]$. Total surplus declines from Π^C to Π^A when the exclusive contract is auctioned to manufacturer A, and manufacturer B is completely foreclosed.

Our model of preference examines the same issues as this literature on exclusive dealing, but does so in the setting of a procurement auction. The results are similar but the auction

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[5] This point is most clearly made in Proposition 5 of the paper by O'Brien and Shaffer (1997), and the subsequent discussion. This finding can also apply to the other dominated equilibria of common distribution by the retailer. Let P_A and P_B be the payments to the manufacturers in such an equilibrium. In order for the retailer to prefer an auction of an exclusive contract, it must be true that $\Pi^B > \Pi^C - P_A - P_B$. Depending on the equilibrium payments to the manufacturers, this condition can be satisfied even though $\Pi^C > \Pi^A$. In particular, this can occur whenever the sum of the equilibrium payments to the manufacturers is such that: $P_A + P_B > \Pi^C - \Pi^B > \Pi^C - \Pi^A > 0$.

setting generates alternative interpretations of the resulting market structure and profits. In the previous literature, the manufacturers make offers of exclusive or non-exclusive contracts which the retailer can only accept or reject. In our model, the buyer can award preference by holding a pre-auction to sell preference to one of the suppliers before holding the procurement auction. The residual bargaining power of the suppliers then derives from their private information about their costs. In the previous literature, an exclusive dealing contract with one manufacturer forecloses completely the manufacturer of the other good. In our model, the preferred supplier merely has a *right-of-first-refusal* for the procurement auction. Thus, the competing suppliers are not completely foreclosed by the sale of preference, and can win the contract when their costs are sufficiently low to bid below the cost of the preferred supplier. This feature is similar to what happens in the model of Aghion and Bolton (1987) which we discuss below.

An important similarity with the previous literature is that preference can increase the profits of the buyer and preferred supplier at the expense of the other suppliers and also reduce social surplus. In the undominated equilibrium of the previous papers, the auction for an exclusive contract divides the combined profits Π^A under the exclusive contract with manufacturer A, with the retailer receiving Π^B and manufacturer A receiving $\Pi^A - \Pi^B$. Thus, the retailer extracts all the profits, $\Pi^C - \Pi^A$, that manufacturer B would have earned in the non-exclusive equilibrium. Manufacturer A then incurs the burden of the social loss from excluding the good of manufacturer B, also equal to $\Pi^C - \Pi^A$. In our model, the buyer extracts rents from the suppliers because the expected profits of the preferred supplier exceed the expected profits of any one of the competing suppliers. The buyer will incur part of the efficiency loss from the preference auction, and will fully incur any loss that might arise from less aggressive bidding by the competing suppliers. However, for large families of cost distributions, the buyer will benefit from selling preference to one of the suppliers.

There is another related literature on vertical foreclosure in which one of two downstream manufacturers acquires or merges with an upstream input supplier in order to foreclose or raise the costs of the other downstream manufacturer. These models have a similar structure to the previous literature on exclusive dealing, but differ in that there are additional competitive effects from the vertical merger on other input suppliers and/or on the ultimate consumers. This literature includes papers by Salop and Scheffman (1983, 1987), Salinger

(1988), Ordover, Saloner, and Salop (1990)[6], Hart and Tirole (1990), Riordan (1998), and Chen (2001). Although these papers posit two or more input suppliers, they can have properties similar to a preference auction in which two buyers compete to be the preferred buyer of a monopoly supplier, the equivalent reverse model from the one in this paper. The reason is that this literature examines the effects of an initial vertical merger between one of the manufacturers and one (or more) of the input suppliers. Depending on different assumptions about the other suppliers, there exist cases in which the integrated manufacturer can prevent a subsequent vertical merger between its rival manufacturer and one of the other suppliers. For these cases, one could envision a pre-auction in which the two manufacturers bid to acquire the most efficient input supplier and thereafter raise the costs of the rival manufacturer. The findings of these papers suggest that such a bidding process would result in a higher combined profit for the integrated firm, and a lower profit for the rival manufacturer. The similarity to the results in this paper is that the integrated firm would be extracting rents from the rival manufacturer. The difference from the results in this paper is that the integrated firm could also be extracting rents from the other input suppliers or from consumers in the final market through higher prices.[7]

Finally, let us briefly discuss the paper by Aghion and Bolton (1987). In their paper, a monopoly buyer contracts with an incumbent supplier when there is some probability of entry by a new supplier with lower costs. The contract specifies one payment to the incumbent supplier for delivery of the good if no entry occurs and a second payment to the incumbent supplier for non-delivery if the buyer obtains the good from the new supplier. The buyer and the incumbent supplier can mutually benefit from this contract because the payments allow them to extract rents from the new supplier when the new supplier enters with lower costs. Our model could be modified to incorporate potential entry by new suppliers. The willingness to pay for preference by each of the existing suppliers participating in the pre-auction could be appropriately adjusted to account for the number and likelihood of new entrants.

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[6] See also the comment and reply by Reiffen (1992) and Ordover, Saloner, and Salop (1992).

[7] In Perry (1978), a monopsonist acquires competitive input suppliers and rationalizes its make-or-buy decision in favor of its internal suppliers. This reduces the competitive rents of the remaining independent suppliers and allows the monopsonist to acquire subsequent suppliers at a lower acquisition price. In this way, the monopsonist can extract rents directly from its suppliers, rather than another rival manufacturer in the downstream stage.

3. Equilibrium Bidding in the Procurement Auction with Preference

The buyer has a value v for a good having a fixed quantity and quality. There are $(n+1)$ suppliers with independent and identical cost distributions for producing the good. The buyer could employ an efficient auction (EA), such as a sealed-bid first-price auction or a second-price auction. As an alternative, we allow the buyer to employ a preference auction (PA) in which one supplier is the preferred supplier (PS) and the other n suppliers are the competing suppliers (CS). In the preference auction, the CS will bid for the contract in a first-price auction, but the PS will then be offered the contract at a price equal to the lowest bid of the CS.[8] The contract will be accepted by the PS if his cost is below this lowest bid of the CS, and rejected otherwise. Thus, preference means that the PS has a *right-of-first-refusal* at the lowest bid of the CS. The PS may be an independent supplier or may be a subsidiary of the buyer. In either case, the PS would not bid against the CS because any bid below all the bids of the CS would only lower the expected price that the PS would be paid by the buyer.[9]

We assume that the ith supplier obtains its cost of production c_i as an independent realization from the same distribution function $G(c)$ with support $[0,1]$, and a positive, continuous density function $g(c)$ over this support. The cost c_i is private information for the ith supplier. For simplicity, we also assume that the value of the good to the buyer exceeds the highest possible cost realization ($v > 1$). Thus, we will examine preference in a symmetric procurement auction with independent private values (costs).

If preference has been awarded, then we will subscript variables related to the preferred supplier (PS) by p and those related to the competing suppliers (CS) by k. We now characterize a symmetric, monotone equilibrium bidding function for the CS, $b(c)$. Assuming that the PS will not reject contracts at a price above his cost, the equilibrium bidding function for each of the CS has to satisfy:

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[8] We assume that the bid of each CS is verifiable so that the buyer can show the PS a signed document with the lowest bid of the CS.

[9] If the PS won, his bid must have been below the lowest bid of the CS. However, the lowest bid of the CS is the price at which the PS would be offered the contract under his *right-of-first-refusal*. Of course, this assumes that any payment for the *right-of-first-refusal* is independent of any bid that the PS might make. See Burguet and Perry (2000) for an alternative model of favoritism.

(1) $c = \arg\max_z \Pi_k[b(z);c] = \arg\max_z (b(z)-c)\cdot[1-G(b(z))]\cdot[1-G(z)]^{n-1}$.

In order for one CS to win the contract, its bid $b(z)$ must be below the cost of the PS and its cost z must be below the costs of the other $(n-1)$ CS. The first-order condition for this problem can be written as

(2) $\dfrac{d\{b(c)\cdot[1-G(b(c))]\cdot[1-G(c)]^{n-1}\}}{dc} = c\cdot\dfrac{d\{[1-G(b(c))]\cdot[1-G(c)]^{n-1}\}}{dc}$. [10]

By integrating (2), an interior solution to (1) must satisfy the following condition

(3a)
$$b(c) = \frac{1}{[1-G(b(c))]\cdot[1-G(c)]^{n-1}}\cdot\int_c^1 z\cdot\frac{d\{[1-G(b(z))]\cdot[1-G(z)]^{n-1}\}}{dz}dz$$
$$= c + \frac{1}{[1-G(b(c))]\cdot[1-G(c)]^{n-1}}\cdot\int_c^1 [1-G(b(z))]\cdot[1-G(z)]^{n-1}dz$$

This condition can be rewritten as the following differential equation

(3b) $b - c = \dfrac{[1-G(b)]}{g(b)} - \dfrac{(n-1)\cdot(b-c)}{b'}\cdot\dfrac{[1-G(b)]}{g(b)}\cdot\dfrac{g(c)}{[1-G(c)]}$. [11]

The optimal bidding function for the CS, $b(c)$, is implicitly defined by (3b). Note that $b(1)=1$, and $b(0)>0$. The problem defined by (1) is dominance solvable when there are only two suppliers ($n = 1$). With only one CS, the bidding function is the best *take-it-or-leave-it* offer from the CS to the buyer who has the option of purchasing the good at the cost of the PS. For the general case of three or more suppliers ($n \geq 2$), the following proposition demonstrates that (3b) defines a unique strictly increasing bidding function $b(c)$ for the CS.

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[10] Monotonicity and symmetry of the bounded bidding function implies that it is also differentiable almost everywhere.

[11] The second-order conditions are satisfied as long as $b(c) > c$ for $c < 1$ at the solution to the differential equation (3b), and then $b'(c) > 0$. Indeed, the first-order condition defines an identity in c, $\left.\dfrac{\partial\Pi_k[b(z);c]}{\partial z}\right|_{z=c} = 0$, so that

Proposition 1: If the density function $g(c)$ is continuously differentiable and bounded away from zero, then there exists a unique strictly increasing bidding function $b(c)$ for the competing suppliers in a preference auction.

Proof: See Appendix

The cost of the PS is a secret random reserve price for the CS. This reserve price is random because it is independently drawn from the same cost distribution. This reserve price is secret because it is not revealed to the CS prior to submitting their bids. One might argue that it is not in the interest of the buyer to maintain the secrecy of the cost of the PS. In particular, the buyer might do better by announcing the cost of the PS, and then running a first-price auction. However, if the cost report is believed by all of the CS, the buyer would have an incentive to report a cost less than the true cost of the PS. If the cost of the PS could not be verified, the equilibrium of this policy would result in exactly the same outcome as the preference auction. Thus, there would be no gain from announcing the cost of the PS as the reserve price.

4. Expected Profits of the Suppliers from the Preference Auction

We can now define the expected profits of both the PS and CS when they compete for the contract in the preference auction. We compare the expected profits of the PS with the CS, and then compare each to the expected profits of a supplier in an efficient auction when no preference is awarded.

In a differentiable monotone equilibrium, incentive compatibility requires that the derivative of the expected profit function $\Pi_k(c)$ of each CS evaluated at cost c satisfies

$$\frac{d}{dc} \left. \frac{\partial \Pi_k[b(z);c]}{\partial z} \right|_{z=c} = \left. \frac{\partial}{\partial z} \frac{\partial \Pi_k[b(z);c]}{\partial z} \right|_{z=c} + \left. \frac{\partial}{\partial c} \frac{\partial \Pi_k[b(z);c]}{\partial z} \right|_{z=c} = 0 .$$ Thus the second-order condition is

simply $\left. \dfrac{\partial}{\partial c} \dfrac{\partial \Pi_k[b(z);c]}{\partial z} \right|_{z=c} = -\dfrac{d[1 - G(b(c))][1 - G(c)]^{n-1}}{dc} \geq 0 .$

$$\frac{d\Pi_k(c)}{dc} = -[1-G(b(c))]\cdot[1-G(c)]^{n-1}.$$

By integrating this expression, we obtain the expected profit function of a CS with cost c:

(4) $\qquad \Pi_k(c) = \int_c^1 [1-G(b(z))]\cdot[1-G(z)]^{n-1} dz \ .$

Similarly, we obtain the expected profit function of the PS with cost c:

(5) $\qquad \Pi_p(c) = \begin{cases} \int_c^1 [1-G(b^{-1}(z))]^n \, dz, & \text{if } c > b(0), \text{ and} \\ [b(0)-c] + \int_{b(0)}^1 [1-G(b^{-1}(z))]^n \, dz, & \text{if } c \le b(0) \end{cases}.$

As a reference point, we can also express the expected profit function of a supplier with cost c in an efficient auction as:

(6) $\qquad \Pi(c) = \int_c^1 G(z)\cdot[1-G(z)]^n dz \ .$

Expressions (4), (5), and (6) are the expected profits of each type of supplier for a given cost c. From these expressions, we can calculate the expected profits of each type of supplier over the distribution of costs:

(7) $\qquad E\Pi_k = \int_0^1 G(c)\cdot[1-G(b(c))]\cdot[1-G(c)]^{n-1} dc,$

(8) $\qquad E\Pi_p = \int_{b(0)}^1 G(c)\cdot[1-G(b^{-1}(c))]^n dc + \int_0^{b(0)} G(c)\, dc \ ,$

(9) $\qquad E\Pi = \int_0^1 G(c)\cdot[1-G(c)]^n dc \ .$

Expressions (7) and (8) are the expected profits when each supplier knows whether it is the PS or one of the CS, but before each supplier knows its cost realization. Expression (9) is the expected profits when each supplier knows that no other supplier will be preferred. The following proposition provides a clear comparison of the expected profits.

Proposition 2: The expected profits of the preferred supplier are greater than its expected profits in an efficient auction; whereas the expected profits of a competing supplier are less than its expected profits in an efficient auction: $E\Pi_p > E\Pi > E\Pi_k$.

<u>Proof of Proposition 2</u>: $E\Pi_p > E\Pi$ follows from the fact that $b^{-1}(c) < c$ for $c < 1$. Similarly,

$E\Pi > E\Pi_k$ follows from the fact that $b(c) > c$ for $c < 1$.   QED

Proposition 2 makes it clear that the PS benefits at the expense of the CS. The next question is whether the buyer can benefit from awarding or selling preference to one of the suppliers.

5.   Pre-Auction to Sell Preference

We assume that the allocation of preference occurs at a point in time earlier than the procurement auction and earlier than the realization of costs by the suppliers. For example, preference could arise before the buyer has determined the quantity and quality of the good, and consequently the suppliers could not know their costs of providing the good. After preference is assigned to one of the suppliers, each supplier learns its cost and the suppliers compete for the contract in the procurement auction discussed in the previous section. For the moment, assume that the PS is an independent supplier who has obtained a *right-of-first-refusal* from the buyer. Irrespective of who obtains the contract, the expected price paid by the buyer is equal to the lowest bid from the CS. Thus, the expected price in the preference auction is

$$(10) \qquad EP_{PA} = \int_0^1 b(c) \cdot n \cdot g(c) \cdot [1 - G(c)]^{n-1} dc \ .$$

If no preference is assigned to any one supplier, then all suppliers compete for the contract in a symmetric first-price auction. Assuming that the buyer must assign the contract to one of the $(n+1)$ suppliers, this first-price auction is an optimal auction for the buyer.[12] Moreover, symmetry among the suppliers ensures that this first-price auction is efficient in that the supplier with the lowest cost will win the contract. The resulting expected price paid by the buyer is

$$(11) \qquad EP_{EA} = \int_0^1 c \cdot (n+1) \cdot n \cdot g(c) \cdot G(c) \cdot [1 - G(c)]^{n-1} dc \ .$$

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[12] In particular, we assume that the buyer cannot set a reserve price. This is a direct corollary of the optimal auction analysis of Myerson (1981). Indeed, when a transaction takes place with probability one, any efficient procedure is an optimal selling or buying procedure.

The expected price with an efficient auction EP_{EA} must be lower than the expected price in the preference auction EP_{PA}.[13] Thus, awarding preference to one supplier without a payment from that supplier is not in the interest of the buyer.[14] However, this conclusion is altered if the buyer can sell preference to one of the suppliers.

As we have seen in the Proposition 2, preference, or a *right-of-first-refusal*, provides one of the suppliers with an advantage in the procurement auction and the other supplier with a corresponding disadvantage. As a result, each supplier has a willingness to pay for preference, derived from the prospect of obtaining the advantage and the desire to avoid the disadvantage. Thus, the buyer could sell preference in a pre-auction to the supplier making the highest bid for preference in the future procurement auction. Since the suppliers do not know their cost realizations at this early stage, they are symmetrically informed and thus have the same willingness to pay for preference. Facing two or more suppliers, the buyer can extract this willingness to pay from one of the suppliers irrespective of the format of the pre-auction or bargaining game with the PS.

The willingness to pay for preference by a supplier is the difference between the expected profits as the PS and expected profits as one of the CS, $E\Pi_p - E\Pi_k$. Thus, when the buyer sells preference, the effective price of the contract paid by the buyer is the difference between the expected price in the preference auction after the contract has been awarded and the payment received from the pre-auction where preference is sold to one of the suppliers. We define this effective price as the net expected price paid by the buyer, NEP_{PA}:

(12) $$NEP_{PA} = EP_{PA} - [E\Pi_p - E\Pi_k].$$

We now ask whether the payment for preference $E\Pi_p - E\Pi_k$ compensates the buyer for the higher expected price EP_{PA} in the preference auction.

Proposition 3: For *n* sufficiently small (*n* = 1), the net expected price when the buyer sells preference is lower than the expected price in an efficient auction: $NEP_{PA} < EP_{EA}$.

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[13] With symmetric suppliers, the expected price in an efficient auction is always lower than the expected price for any auction mechanism where the buyer purchases with probability one, but which does not allocate the contract to the lowest cost supplier in all cases.

[14] This is consistent with the findings of Bikhchandani, Lippman, and Ryan (2005) for second-price auctions.

Proof of Proposition 3: By adding and subtracting $E\Pi$, the difference between the net expected price and the expected price in an efficient auction can be expressed as three terms:

$$NEP_{PA} - EP_{EA} = [EP_{PA} - E\Pi_p] + [E\Pi - EP_{EA}] - [E\Pi - E\Pi_k] \ .$$

The third term in this expression is negative from Proposition 2 because $E\Pi > E\Pi_k$. In order to examine the other two terms, denote $Y_{(i;m)}$ as the ith order statistic (from low to high) of m realizations of costs. Since the expected price in an efficient (second-price) auction EP_{EA} equals the expected value of $Y_{(2;n+1)}$, the expected profit $E\Pi$ of a supplier when no preference is awarded is the expected value of $[Y_{(2;n+1)} - Y_{(1;n+1)}]/(n+1)$. Thus, the second term is negative because $E\Pi - EP_{EA} = -[n \cdot EY_{(2;n+1)} + EY_{(1;n+1)}]/(n+1) < 0$. Now consider the first term. Note that the expected profits of the PS can be expressed as

$$E\Pi_p = \int_0^1 \int_{\max\{b^{-1}(c),0\}}^1 (b(x) - c) \cdot n \cdot g(x)[1 - G(x)]^{n-1} dx \cdot g(c) \, dc, \qquad \text{where} \qquad \max\{b^{-1}(c),0\}$$

indicates that the lower limit is 0 whenever $b^{-1}(c)$ does not exist, that is, whenever $b(0) > c$. Also note that $EY_{(1;n)} = \int_0^1 c \cdot n \cdot g(c)[1 - G(c)]^{n-1} dc$. The second term can now be simplified as follows:

$$
\begin{aligned}
EP_{PA} - E\Pi_p &= \int_0^1 (b(c) - c) \cdot n \cdot g(c)[1 - G(c)]^{n-1} dc + EY_{(1;n)} - E\Pi_p \\
&= \int_0^1 \int_0^1 (b(x) - x) \cdot n \cdot g(x)[1 - G(x)]^{n-1} dx \cdot g(c) dc + EY_{(1;n)} \\
&\quad - \int_0^1 \int_{\max\{b^{-1}(c),0\}}^1 (b(x) - c) \cdot n \cdot g(x)[1 - G(x)]^{n-1} dx \cdot g(c) dc \\
&= \int_0^1 \int_0^1 (c - x) \cdot n \cdot g(x)[1 - G(x)]^{n-1} dx \cdot g(c) dc + EY_{(1;n)} - C \\
&= E[c] - C \ ,
\end{aligned}
$$

where $C = -\int_{b(0)}^1 \int_0^{b^{-1}(c)} (b(x) - c) \cdot n \cdot g(x)[1 - G(x)]^{n-1} dx \cdot g(c) dc > 0$. The first equality simply adds and subtracts $EY_{(1;n)}$. The second equality substitutes the equivalent double integral with a change in the variables of integration for the first integral so that it can be combined with

the double integral defining $E\Pi_p$. The third equality rearranges these two double integrals creating the term C. The fourth equality simply cancels $EY_{(1;n)}$. Thus,

(13) $NEP_{PA} - EP_{EA} = E[c] - C - \dfrac{n \cdot E[Y_{(2;n+1)}] + E[Y_{(1;n+1)}]}{n+1} - [E\Pi - E\Pi_k]$.

When $n = 1$, $E[c] = [EY_{(2;2)} + EY_{(1;2)}]/2$, so that $NEP_{PA} - EP_{EA} = -C - [E\Pi - E\Pi_k] < 0$.

QED

Proposition 3 is completely general in terms of cost distributions, but it only guarantees that the buyer prefers selling preference for the case when there are two suppliers ($n = 1$). Since two suppliers is only a sufficient condition, the proof clearly suggests that there must be cost distributions for which this result would also be true for a larger number of suppliers ($n > 1$). Indeed, the next proposition demonstrates that there exists a wide class of cost distributions for which the buyer prefers selling preference irrespective of the number of potential suppliers.

Proposition 4: If the inverse hazard rate, $[1 - G(c)]/g(c)$, is decreasing and convex, then for any number of suppliers $n \geq 1$, the net expected price when the buyer sells preference is lower than the expected price in an efficient auction: $NEP_{PA} < EP_{EA}$.

Proof of Proposition 4: By adding and subtracting $E\Pi$, the difference between the net expected price and the expected price in an efficient auction can be expressed as three terms:

$$NEP_{PA} - EP_{EA} = [EP_{PA} - EP_{EA}] - [E\Pi_p - E\Pi] - [E\Pi - E\Pi_k] \ .$$

Let $b_{EA}(c)$ be the bidding function in an efficient (first-price) auction with ($n + 1$) symmetric suppliers. The expected price for this efficient auction can be decomposed as follows

$$EP_{EA} = \int_0^1 b_{EA}(c) \cdot (n+1) \cdot g(c)[1 - G(c)]^n dc$$

$$= \int_0^1 b_{EA}(c) \cdot n \cdot g(c)[1 - G(c)]^n dc - \int_0^1 b_{EA}(c) \cdot n \cdot g(c)G(c)[1 - G(c)]^n dc$$

$$+ \int_0^1 b_{EA}(c) \cdot g(c)[1 - G(c)]^n dc \ .$$

From the proof of Proposition 3, the expected profits of the PS can be bounded below by

$$E\Pi_p = \int_0^1 \int_{\max\{b^{-1}(c),0\}}^1 (b(x)-c)\cdot n\cdot g(x)[1-G(x)]^{n-1}dx\cdot g(c)\,dc$$

$$> \int_0^1 \int_c^1 (b(x)-c)\cdot n\cdot g(x)[1-G(x)]^{n-1}dx\cdot g(c)\,dc$$

$$= \int_0^1 \int_c^1 b(x)\cdot n\cdot g(x)[1-G(x)]^{n-1}dx\cdot g(c)\,dc \; - \; EY_{(1;n+1)}/(n+1)$$

$$= \int_0^1 b(c)\cdot n\cdot g(c)[1-G(c)]^{n-1}G(c)\,dc \; - \; EY_{(1;n+1)}/(n+1)\;.$$

The inequality arises from the fact that $c > \max\{b^{-1}(c),0\}$, and the last equality is obtained by changing the order of integration. Substituting these two expressions into the difference between the net expected price and the expected price in an efficient auction, and rearranging terms, we find that

$$NEP_{PA} - EP_{EA} \; < \; \int_0^1 [b(c)-b_{EA}(c)]\cdot n\cdot g(c)[1-G(c)]^{n-1}dc$$

$$- \int_0^1 [b(c)-b_{EA}(c)]\cdot n\cdot g(c)G(c)[1-G(c)]^{n-1}dc$$

$$- \left(\int_0^1 b_{EA}(c)\cdot g(c)[1-G(c)]^n dc \; - \; EY_{(1;n+1)}/(n+1) \; - E\Pi \right)$$

$$- \; [E\Pi - E\Pi_k]\;.$$

The third term on the right-hand side is zero. Combining the first two terms, we obtain a simple upper bound on the difference:

$$(14) \qquad NEP_{PA} - EP_{EA} \; < \; \int_0^1 \big[b(c)-b_{EA}(c)\big]\cdot n\cdot g(c)[1-G(c)]^n dc \; - \; [E\Pi - E\Pi_k]$$

As shown in the proof of Proposition 2, $E\Pi - E\Pi_k > 0$. Thus, whether the buyer benefits from selling preference depends on the integral of the difference in the bidding functions of the CS before and after preference is awarded. In particular, the buyer clearly benefits from selling preference if the CS bid the same or bid uniformly more aggressively in the preference auction than in the efficient first-price auction. Arozamena and Weinschelbaum (2004) and Porter and Shoham (2005) have examined the bidding behavior of n symmetric suppliers in a first-price auction when another supplier has a *right-of-first-refusal*. Both papers have shown that $[b(c)-b_{EA}(c)]$ is negative for all c if the inverse hazard rate of the cost distribution is decreasing and convex. Thus, this condition on the inverse hazard rate is sufficient for our result that $NEP_{PA} - EP_{EA} < 0$. QED

Proposition 4 provides insight into how the sale of preference extracts rents from the CS. If the inverse hazard rate of the cost distribution is linearly decreasing, then the CS do not alter their bidding function after preference is awarded to one supplier. In this case, the buyer extracts $E\Pi - E\Pi_k > 0$ in rents from the CS. An example of such a cost distribution is the power family $G(c) = 1 - [1 - c]^t$, where $t > 0$. If the inverse hazard rate is decreasing and convex, then the buyer extracts additional surplus from the CS because they also bid more aggressively after preference is awarded to one supplier.

6.   The Joint Surplus and Vertical Merger of the Buyer and the Preferred Supplier

Under the conditions of Proposition 3 or 4, the buyer benefits from selling preference. The remaining question is whether the buyer and the PS can jointly benefit from preference. Simply stated, we ask whether the gains to the buyer from the lower net expected price exceed the losses to the PS after paying for preference. The answer to this question determines whether the buyer and one supplier have a mutual incentive to merge vertically. Obviously, the integrated firm would employ preference for its internal supplier, so that the preference auction with the CS would be unchanged. If the joint surplus increases with a vertical merger, then preference for the internal supplier would extract rents from the CS which could then be divided between buyer and PS. Thus, preference would not necessarily arise solely from the buyer holding a pre-auction for preference to extract the difference between the expected profits from becoming the PS rather than remaining one of the CS. Instead, preference could also arise from a vertical merger in which the buyer and one supplier bargain to divide the joint gains created by preference.

Using a second-price auction, Bikhchandani, Lippman, and Ryan (2005) found that the joint surplus was unchanged by awarding a *right-of-first-refusal*. As such, there was no incentive for the buyer and one supplier to merge vertically. The following proposition demonstrates that this result does not apply to the first-price auction used in this paper to define the preference auction.

Proposition 5: If the number of suppliers is sufficiently small (Proposition 3) or if the inverse hazard rate of the cost distribution is decreasing and convex (Proposition 4), the joint surplus of the buyer and the preferred supplier is higher with preference:

$$v + E\Pi_p - EP_{PA} > v + E\Pi - EP_{EA}.$$

<u>Proof of Proposition 5</u>: The payment for preference $E\Pi_p - E\Pi_k$ is simply a transfer from the PS to the buyer, so the expected prices, EP_{PA} and EP_{EA}, are the net expected payments to the CS. By adding and subtracting $E\Pi_k$, we can rewrite the difference in joint surplus:

$$[v + E\Pi_p - EP_{PA}] - [v + E\Pi - EP_{EA}]$$
$$= EP_{EA} - \{EP_{PA} - [E\Pi_p - E\Pi_k]\} - [E\Pi - E\Pi_k]$$
$$= [EP_{EA} - NEP_{PA}] - [E\Pi - E\Pi_k] > 0$$

The last inequality follows from (13) in the proof of Proposition 3 or (14) in the proof of Proposition 4. QED

The result in Proposition 5 is derived from a first-price auction and differs from the findings of Bikhchandani, Lippman, and Ryan (2005) that were derived from a second-price auction. In a second-price auction, it is a dominant strategy for a supplier to bid its cost with or without preference, that is, with or without a secret random reserve price. As such, the CS would not bid more (or less) aggressively in the presence of a PS. Thus, the price will equal $Y_{(2;n)}$, instead of $Y_{(2;n+1)}$ and the expected price paid by the buyer will increase by an amount equal to $EY_{(2;n)} - EY_{(2;n+1)} = 2 \cdot [EY_{(3;n+1)} - EY_{(2;n+1)}]/(n+1)$. With a second-price auction, the PS would obtain the contract at a price equal to the third lowest cost when its cost is either the lowest or the second lowest. Without preference, a supplier would obtain the contract at a price equal to the second lowest cost when its cost is the lowest. Trivially, we can also include the case in which a supplier would obtain the contract at that same price when its cost is the second lowest, earning zero profit. These two events have a probability of $(n + 1)$. Thus, the increase in expected profits from becoming the PS is exactly equal to the increase in the expected price paid by the buyer. This is the result of Bikhchandani, Lippman, and Ryan (2005).

Unlike a second-price auction, the CS in a first-price auction may bid more aggressively in the presence of a PS. The expected price paid by the buyer increases because one less supplier is bidding for the contract, but this increase is moderated because of the more aggressive bidding by each of the CS. Second, a first-price auction results is a different distortion in the allocation of the contracts to the PS than a second-price auction. For a given realization x of the lowest cost of the CS, the PS wins the contract with probability one in a first-price auction if its cost c is lower than the bid $b(x)$ of the CS. However, even conditional on $b(x) > c$, there is a positive probability in a second-price auction that the second lowest cost of the CS is lower than c. Of course, when $b(x) < c$, the PS may still win the contract in a second-price auction, but not in a first-price auction. Propositions 3 and 4 have identified sufficient conditions under which the effects from the bidding of the CS dominate the effects from the allocative distortion.

One implication of Proposition 5 is that a vertical merger could occur at a point when the buyer could not realistically hold a pre-auction for preference. For example, if the joint surplus was greater with preference, there would be an incentive for a vertical merger between the buyer and a sole existing supplier, even though there is uncertainty about the number of new suppliers who will subsequently enter the market and participate in the procurement auction. The results of Proposition 5 would remain qualitatively unchanged as long as the buyer and the sole existing supplier agree on the likelihood of different realizations on the number of new suppliers at the time that they negotiate over the division of the gains from the vertical merger.

7. Concluding Remarks

The results in this paper provide another illustration of how vertical contracts can be employed to increase the profits of a buyer by extracting rents from some or all of its suppliers. By awarding preference to one supplier, the buyer can commit to a secret reserve price that reduces the expected profits of the suppliers. The buyer can then extract the rents that would accrue to the PS by holding a pre-auction for preference. As a result, the buyer can lower the net expected price paid for the good. After paying for preference, the expected profits of the PS are the same as the CS. However, preference increases the joint surplus of the buyer and PS. Thus, preference can arise from either a pre-auction of a *right-of-first-refusal*, or a vertical merger agreement to share the gains from preference. Either way, preference is a simple mechanism for

increasing the net surplus of the buyer. Preference requires only a simple contract between the buyer and the PS for which neither has an incentive to breach. The buyer has no incentive to take delivery from one of the CS at a price higher than the cost of the PS, and the PS has a choice of accepting or rejecting the contract at the lowest bid from the CS. In other words, the buyer and PS need make no additional commitments, other than their original contract creating a *right-of-first-refusal* or the integrated firm.

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Appendix: Proof of Proposition 1

We need only proving that the initial value problem (3b) with $b(1) = 1$ has a unique, strictly monotone solution whenever $g(c)$ is continuously differentiable and bounded away from zero.

Recall that $J(x) \equiv \dfrac{1 - G(x)}{g(x)}$. Let us write the initial value problem (3b) with $b(1) = 1$ as:

$$b' = \phi(b, c), \quad b(1) = 1$$

where

$$\phi(b, c) \equiv \frac{(n-1)(b-c)g(c)J(b)}{J(c)[J(b) - (b-c)]}.$$

We cannot readily apply standard existence results to this initial value problem, since $\phi(b, c)$ is not well defined around (1,1). The trouble is with the denominator of that expression. Let us define $h(c)$ as the inverse of the function $h^{-1}(x) = x - J(x)$. Note that the denominator in the definition of ϕ is 0 at any point $(c, h(c))$. At any point (c, b) with $b < h(c)$, the denominator is positive and ϕ is well defined. Also, note that $h(c)$ is an increasing, continuous function and $h(1) = 1$. Finally, note that $h'(c)$ is bounded above by 1. Indeed, $\dfrac{dh^{-1}(x)}{dx} = 1 - J'(x) > 1$.

Instead of considering our original initial value problem, consider the following sequence of initial value problems. Let $\{c_m\}$ be an increasing sequence of numbers that converge to 1. Define the initial value problem

$$\hat{b}_m' = \phi(\hat{b}_m, c), \quad \hat{b}_m(c_m) = c_m .$$

The function $\phi(b, c)$ and also $\dfrac{\partial \phi(b, c)}{\partial b}$ are continuous for $c \in [0, 1)$, and $b \in [0, h(c))$, if g is positive and continuously differentiable. Thus, there exists a unique solution $\hat{b}_m(c)$ to that initial value problem in some interval $(c_m\text{-T}, c_m]$, for some positive number T. We note that the solution $\hat{b}_m(c) \in [c, h(c))$. Indeed, $h'(c)$ is bounded above by 1 but $\phi(b, c) \to \infty$ as $b \to h(c)$. This shows that indeed $\hat{b}_m(c) < h(c)$. We conclude that the unique solution $\hat{b}_m(c)$ exists in $[0, c_m]$.

Then, define the monotone, continuous function

$$b_m(c) = \begin{cases} \hat{b}_m(c) & \text{if } c < c_m \\ c & \text{if } c \geq c_m \end{cases}$$

in the interval $[0,1]$.

Lemma 1: The sequence $b_m(c)$ is non decreasing in m for all $c \in [0,1]$. Also, $b_m(c) \in [c, h(c))$.

<u>Proof</u>: Consider the functions b_m and b_{m+1}. At the point c_{m+1}, $b'_m(c_{m+1}) = 1$, whereas the left derivative of $b_{m+1}(c_{m+1})$ is 0. Thus, in an open interval to the left of c_{m+1}, $b_{m+1}(c) > b_m(c)$. We show next that the continuous functions $b_{m+1}(c)$ and $b_m(c)$ do not cross. Assume that for some $c < c_{m+1}$, $b_{m+1}(c) = b_m(c)$. If $c > c_m$, then again $b'_m(c) = 1$, and $b'_{m+1}(c) = 0$. The latter follows from the fact that $b_m(c) = c$ in this interval, and then if $b_{m+1}(c) = b_m(c) = c$, the numerator in the definition of ϕ is zero. Now, if $c \leq c_m$, then $b_{m+1}(c') = b_m(c')$ for all c'<c. Indeed, both b_m and b_{m+1} would be the solution to the same, new initial value problem defined by (3b) with $b(c') = b_m(c)$ for $c' = c$. Then we conclude that the functions do not cross, and then indeed $b_{m+1}(c) \geq b_m(c)$ for all c.

To prove the second part of the lemma, we first show that b_m does not cross the function c. In the interval $[c_m, 1]$, the two functions coincide. As we have shown above, the left derivative of b_m at c_m is zero, and then b_m is above c at the left of c_m. If at some $c < c_m$, $b_m(c) = c$, then once again the derivative of b_m at c is 0, and the derivative of c is 1. Thus, we conclude that b_m does not cross c. QED

A consequence of the first part of Lemma 1 is that $\{b_m(c) - b_{m-1}(c)\} \to 0$ for all $c \in [0,1]$. Let $b(c) = \lim_{m \to \infty} b_m(c)$. By the second part of Lemma 1, $J(b_m(c)) + c > b_m(c) \geq c$, so that $b'_m(c) > 0$. Pointwise convergence implies that $b(c)$ is monotone, but does not guarantee continuity of $b(c)$. The next lemma shows that $b(c)$ is indeed continuous.

Lemma 2: $b(c)$ is continuous. That is, b_m converges uniformly on $[0,1]$.

<u>Proof:</u> Assume that $b(c)$ is discontinuous at a point $c'<1$. Let $b_- = \lim_{c\uparrow c'} b(c)$ and $b_+ = \lim_{c\downarrow c'} b(c)$. Also, let $b_+ - b_- = \alpha > 0$. Since $b_m(c) \to b(c)$ and $b_m(c)$ is a non decreasing sequence of continuous, monotone functions, then $\forall \varepsilon_1, \delta_1 > 0, \exists m$ such that $b_m(c + \delta_1) - b_m(c - \delta_1) > \alpha - \varepsilon_1$, in which case $\exists c \in (c'-\delta_1, c'+\delta_1)$ such that $b'_m(c) = \phi(b_m(c), c) \geq \dfrac{\alpha - \varepsilon_1}{2\delta_1}$. For δ_1 small enough (consider only δ_1 smaller than $(1-c')/3$), the right hand side is as large as desired. That is, $\forall \varepsilon_1, \delta_1 > 0, \exists m, c \in (c'-\delta_1, c'+\delta_1)$ such that $\phi(b_m(c), c)$ is as large as desired. Or equivalently, $b_m(c)$ is as close to $h(c)$ as desired and $\big[J(b_m(c)) - (b_m(c) - c) \big]$ is as close to 0 as desired. That shows that, $\forall \varepsilon > 0, \exists m, c, \delta_1$ such that

$$\phi(b_m(c), c) > \frac{g(c)(n-1)J(b_m(c))}{J(c)\varepsilon},$$

and also $\varepsilon > h(c) - b_m(c)$.

Let $\delta_2 = \dfrac{1-c'}{3}$. Since J is non increasing and $g(c)$ is bounded below, $g(c) > A > 0$, the above inequality implies

$$\phi(b_m(c), c) > \frac{A(n-1)J(b_m(c'+\delta_1 + \delta_2))}{J(c')\varepsilon} \equiv \Delta(\varepsilon).$$

For ε small enough, that expression is larger than 1, and so larger than $h'(c)$. (Recall that $h'(c)$ is bounded above by 1.) In other words, for values $c''>c$, $h(c) - b_m(c)$ does not get larger. Thus, we can conclude that the inequality above holds for any c'' in the interval $(c, c'+\delta_1 + \delta_2)$. Take ε so that $\Delta(\varepsilon)\delta_2 > \varepsilon + 2\delta_2$, and since $\delta_2 > \delta_1$, $\varepsilon > h(c) - b_m(c)$, and $h'(c') < 1$, we conclude that $b_m(c) > h(c)$. This contradicts Lemma 1 and then shows that indeed $b(c)$ is continuous at $c<1$.

For $c=1$, continuity is a straightforward consequence of our construction of the sequence of functions $b_m(c)$. QED

We now show that the limit, $b(c)$, is a solution to the initial value problem (3b) and is strictly monotone. To that end, we first show that b_m' converges uniformly in $[0,1)$.

Lemma 3: b_m' converges uniformly in $[0,1)$.

Proof: First notice that b_m' is well defined except at c_m. Now, for any point c in $[0,1)$, let M be such that $c < c_M$. For all $m > $ M, $b_m'(c) = \phi(b_m(c),c)$. According to Lemma 1, $b_m(c)$ is non-decreasing in m, $b_m(c) \geq c$, and also we have that $\phi(b,c)$ is monotone increasing in $b \geq c$. Thus, $b_m'(c)$ is increasing in m and bounded above by $\phi(b(c),c)$. From Lemma 1 we also know that $b(c) \leq h(c)$, but we have to show that $b(c) < h(c)$ in order to guarantee that $\phi(b(c),c)$ (the upper bound) is finite. To this end, note that if $b(c) = h(c)$ then for m large enough $b_m'(c) = \phi(b_m(c),c)$ is arbitrarily large and at the same time $b_m(c)$ arbitrarily close to $h(c)$. Since $h'(c)$ is bounded above by 1 we would have that for some m, b_m would cross h, which would contradict Lemma 1.

Thus, we can guarantee that $b_m'(c)$ converges. Let $b'(c)$ denote that limit, and assume that it is not continuous. That is, for some c, say $c = 1-a$ for some a in $(0,1]$, there exits ε such that

$$\forall \delta, \exists c' \; s.t. \left|c'-c\right| < \delta \; and \; \left|b'(c') - b'(c)\right| > \varepsilon.$$

Then, fix $\delta < a$ so that

i) $\qquad \forall c'' \; s.t. \left|c''-c\right| < \delta, \; \left|\phi(b(c''),c'') - \phi(b(c),c)\right| < \varepsilon/5$

and consider the corresponding c'. We know that δ exists since ϕ is continuous (and then uniformly continuous) in the compact set $\left\{(x,y) \in R^2 ; x \in [0,1-a], y \in [x,b(x)]\right\}$ and $b(c)$ is uniform continuous in $[0,1-a]$. Next, fix M such that for all $m > $ M,

ii) $\qquad \left|b_m'(c) - b'(c)\right| < \varepsilon/5$,

iii) $\qquad \left|b_m'(c') - b'(c')\right| < \varepsilon/5$

iv) $\qquad \forall c'', \; \left|\phi(b_m(c''),c'') - \phi(b(c''),c'')\right| < \varepsilon/5$.

We know that M exists since b_m' converges pointwise to b' -(ii) and (iii)-, b_m converges uniformly to b, and again ϕ is uniformly continuous -iv-. Now,

$$|b'(c') - b'(c)| \leq |b'(c') - b_m'(c')| + |b'(c) - b_m'(c)| + |b_m'(c') - b_m'(c)|,$$

and

$$|b_m'(c') - b_m'(c)| = |\phi(b_m(c'),c') - \phi(b_m(c),c)| \leq$$
$$|\phi(b_m(c'),c') - \phi(b(c'),c')| + |\phi(b_m(c),c) - \phi(b(c),c)| + |\phi(b(c'),c') - \phi(b(c),c)|.$$

Thus, from (i), (ii), (iii), and (iv), we conclude that $|b'(c') - b'(c)| < \varepsilon$, and this contradiction proves that $b'(c)$ is continuous. That proves the lemma. QED

We are now ready to show that $b(c)$ is a solution to (3b). F irst note that $b_m(1) = 1$ for all m, so that indeed $b(1) = 1$. Now, take a new sequence of initial value problems defined by (3b) and $b_m(c_m) = b(c_m)$. Note that $b(c)$ is the solution to each of these initial value problems in the corresponding intervals $[0,c_m]$. Indeed,

$$\lim_{\varepsilon \to 0} \frac{b(c) - b(c - \varepsilon)}{\varepsilon} = \lim_{\varepsilon \to 0} \lim_{m \to \infty} \frac{b_m(c) - b_m(c - \varepsilon)}{\varepsilon} =$$
$$\lim_{m \to \infty} \lim_{\varepsilon \to 0} \frac{b_m(c) - b_m(c - \varepsilon)}{\varepsilon} = \lim_{m \to \infty} \phi(b_m(c),c) = \phi(b(c),c),$$

where the second equality follows from uniform convergence of $b_m'(c)$ and the fourth from the fact that $\phi(b(c),c)$ is continuous in the interval $[0,c_m]$.

Now $\{c_m, b(c_m)\} \to \{1,1\}$, and thus $b(c)$ is indeed the solution to our initial value problem.

Second, we need to show that $b(c)$ is strictly monotone. Assume this is not true, so that at some point $c < 1$, we have that $b'(c) = 0$. That can only happen when $b(c) = c$, in which case the function $b(c)$ would cross the 45° line at c, so that for some $c'>c$, $b(c') < c'$. This contradicts Lemma 1. Thus, $b(c)$ is strictly monotone. QED