A macroeconomic foundation for the Nelson and Siegel class of yield curve models

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Abstract

Yield curve models of the Nelson and Siegel (1987) class have proven themselves popular empirical tools in finance and economics, but they lack a formal theoretical justification. Hence, this article uses a multifactor version of the Cox, Ingersoll and Ross (1985*a*) continuous-time general-equilibrium economy to derive a macroeconomic foundation for a theoretically-consistent version of the Nelson and Siegel class of yield curve models. It is established that the level and shape of the yield curve as represented by NS models may be explained succinctly in terms of expectations of inflation and real output growth within an underlying economic model. This theoretically-rigorous yet parsimonious and intuitive framework is applicable as a macro-finance tool, and the application in this article provides a ready interpretation of a series of empirical results from the macro-finance literature that relate the level and slope of the yield curve to output growth and inflation.

JEL: E43, E31, E32

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1 Introduction

This article develops a formal macroeconomic foundation for the popular Nelson and Siegel (1987) (hereafter NS) class of yield curve models. Specifically, it establishes that the level and shape of the yield curve as represented by NS models may be explained succinctly in terms of expectations of inflation and real output growth within an underlying economic model.

The initial motivation is to justify the NS approach to yield curve modelling from a theoretical perspective. However, an equally-important contribution discussed subsequently below is to the field of macro-finance, a growing literature that explores the interlinkages, relationships, and information in common between financial markets and economic variables.

The essence of the NS approach, as will be detailed in section 2, is to fit yield curve data at a given point in time with a parsimonious linear combination of simple functions of time-to-maturity; i.e Level, Slope, and Bow (or Curvature) components. This simple structure makes NS models easy to estimate, and the estimated coefficients provide an intuitive quantitative summary of the level and shape of the yield curve. Not surprisingly then, NS models have proven very popular empirical tools in finance and economics. They are applied extensively by academics and practitioners across many markets and countries for routine yield curve analysis and zero-coupon

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interest rate estimation, where they have typically proven to be comparable or superior to more complex approaches to yield curve modelling.[1]

However, more recent applications have pushed beyond the simple curve-fitting for which NS models were originally developed. For example, Fabozzi, Martellini and Priaulet (2005), Diebold and Li (2006), and Diebold, Rudebusch and Aruoba (2006) use estimated NS coefficients directly within a time-series/forecasting context without addressing the theoretical criticism in Filopović (1999 and 2000) that NS models cannot be intertemporally consistent and arbitrage free. These technical issues are rectified in Sharef and Filipović (2004), Krippner (2006), and Christensen, Diebold and Rudubusch (2007) with versions of the NS model that specify Gaussian dynamics for the NS components, but a fundamental issue still remains: are NS components with Gaussian dynamics an appropriate representation for a yield curve model? Addressing this question with a sound theoretical basis is essential if the trend continues for NS models to be applied more widely, because it will suggest when their application is likely to be appropriate relative to more complex models of the yield curve, or highlight where appropriate modifications to the NS approach should be made.

Given the yield curve is ultimately a reflection of the underlying economy, a sensible starting point for a theoretical justification to the NS class of yield curve models is within an economic model. Hence, in section 3, the derivation of a macroeconomic foundation begins with the specification of a generic multifactor version of the standard continuous-time general-equilibrium model of the economy from Cox, Ingersoll and Ross (1985a). The yield curve and its dynamics derived from that economic model under rational expectations are then compared explicitly to the yield curve and dynamics of the intertemporally-consistent and arbitrage-free version of NS models from Krippner (2006); i.e the so-called augmented NS (hereafter ANS) model. This comparison shows that the ANS Level component equates to within a parametric term premium to long-term expected inflation plus potential growth. The ANS Slope and Bow components reflect to within a parametric term premium function the expected mean reversion of the current levels of inflation and output growth back to their anticipated long-run levels.

The ANS framework (i.e the ANS model with its macroeconomic foundation) has an obvious connection to the field of macro-finance, given that it offers a theoretically-rigorous yet parsimonious and intuitive connection between the yield curve, output growth, and inflation. As a simple and practical illustration of its application as a macro-finance tool, the ANS framework is used in this article to provide ready interpretations of relationships between the yield curve, output growth, and inflation that have been well established empirically within the literature. For example, heuristically-justified ordinary least squares (OLS) regressions have established a strong relationship between the current slope of the yield curve (typically measured as the 10-year government bond yield less the 3-month Treasury bill rate) and future output growth, a moderate relationship between the current slope of the yield curve and future inflation, and a cointegrating relationship between term interest rates (typically ranging from 1 to 5 years) and inflation.[2] Another empirical investigation, which is closely related to this article because it uses NS components as latent factors to represent the yield curve, is the vector autoregressive (VAR) model of Diebold

[1] Ten of twelve central banks in the Bank for International Settlements (2005) survey use NS models or the Svensson (1995) extension, and recent examples of the application of NS models not otherwise referenced in this article include Jankowitsch and Pilcher (2004), Steeley (2004), Diebold, Ji and Li (2005), and Gürkaynak, Sack and Wright (2006). Dahlquist and Svensson (1996), Bliss (1997), and Ioannides (2003) are examples that report favourable empirical comparisons with other approaches to modelling the yield curve.

[2] Berk (1998) provides a useful survey. Recent examples reporting these results include: Hamilton and Kim (2002), Bordo and Haubrich (2004), Nakaota (2005) and Paya, Matthews and Peel (2005) for yield curve/output relationships; Estrella, Rodrigues and Schich (2003) for yield curve/inflation relationships; and Fahmy and Kandi (2003) and Lai (2004) for interest rate/inflation cointegration. Estrella (2004) pp. 722-723 notes that the various justifications advanced for these empirical relationships are generally informal or heuristic: e.g real business cycles, countercyclical monetary policy, and life-cycle consumption to justify yield curve/output relationships; and the Fisher hypothesis with assumed constant or stationary real interest rates to justify yield curve/inflation relationships and interest rate/inflation cointegration.

et al. (2006).[3] The latter confirms the relationships from the yield curve to the macroeconomic variables already noted, but also establishes their reverse; i.e the long-maturity level of the yield curve responds to changes in inflation (and also to changes in real activity, as measured by capacity utilisation), and the slope of the yield curve responds to changes in real activity and inflation.

To show how the ANS framework explains the empirical results above, section 4 uses it to explicitly derive theoretical econometric relationships between the yield curve, output growth, and inflation analogous to the OLS regressions and the Diebold et al. (2006) VAR application mentioned above. Those derivations embed the two-way yield curve/macroeconomic empirical relationships already noted, but with several predicted extensions, and also provide a transparent theoretical basis for why the data should support those empirical relationships. Estimating the derived econometric relationships using 54 years of United States data in section 5 provides empirical support for the predictions from the ANS framework, and also highlights that occasional and highly persistent changes to term premia have played a very influential role with historical relationships between the yield curve, output growth, and inflation.

Section 6 concludes, and then briefly discusses several potential applications for the ANS framework.

2 The NS and ANS models of the yield curve

This section introduces the key aspects of the NS class of yield curve models and the intertemporally-consistent and arbitrage-free ANS model from Krippner (2006) that are relevant to this article.

Models of the NS class are generally linearly equivalent to the following specification:

$$f(t,m) = \sum_{n=1}^{3} \beta_n^{\mathrm{NS}}(t) \cdot g_n(\phi, m) \tag{1}$$

where $f(t,m)$ is the (instantaneous continuously-compounding) forward rate curve as at time t as a function of time-to-maturity m, $\beta_n^{\mathrm{NS}}(t)$ are three coefficients at time t that are associated with the three time-invariant functions of time-to-maturity $g_n(\phi,m)$, and the latter are defined as $g_1(\phi,m) = 1$, $g_2(\phi,m) = -\exp(-\phi m)$, and $g_3(\phi,m) = -\exp(-\phi m)(-2\phi m + 1)$, where ϕ is a positive constant parameter that governs the rate of exponential decay.[4] Figure 1 illustrates these functions, which are named the Level, Slope, and Bow modes based on their intuitive shapes, and hence $\beta_1^{\mathrm{NS}}(t)$, $\beta_2^{\mathrm{NS}}(t)$, and $\beta_3^{\mathrm{NS}}(t)$ are the Level, Slope, and Bow coefficients. The (zero-coupon continuously-compounding) interest rate curve, at time t and as a function of time-to-maturity m, is then $R(t,m) = \frac{1}{m}\int_0^m f(t,m)dm$ and that is sometimes used as a direct link to estimating the NS coefficients from derived zero-coupon yield curve data. Alternatively, the associated discount function $\exp[-m \cdot R(t,m)]$ provides the basis for estimating the NS coefficients at time t by fitting the market-quoted prices and cashflows of the interest rate securities that compose the yield curve at time t. In either case, the parameter ϕ is generally fixed at a value that provides an appropriate fit to the overall series of historical yield curve data.

[Figure 1 here]

The ANS model from Krippner (2006) adds the minimum number of additional parameters required to address the theoretical criticism by Filopović (1999 and 2000) that NS models cannot be intertemporally-consistent and arbitrage free. Specifically, "the expected path of the short rate"

[3] Diebold, Piazzesi and Rudebusch (2005) provides a summary of Diebold et al. (2006) and suggested extensions in the context of recent advances in the macro-finance literature.

[4] The original NS article notes that the components are a solution to a second-order differential equation with equal roots. The alternative NS models of Svensson (1995), Bliss (1997), and Mansi and Phillips (2001) are analogous to the specification in equation 1, but they also contain exponential terms with different decay rates.

for the ANS model is defined as:

$$E_t\left[r\left(t+m\right)\right]=\sum_{n=1}^{3}\lambda_n\left(t\right)\cdot g_n\left(\phi,m\right) \tag{2}$$

where E_t is the expectations operator conditional upon information available at time t, $E_t\left[r\left(t+m\right)\right]$ is the expected path of the (instantaneous continuously-compounding) short rate as at time t and as a function of future horizon m (so $t+m$ is a future point in time), and $\lambda_n\left(t\right)$ are three latent state variables associated with the NS functions $g_n\left(\phi,m\right)$. The dynamics for $E_t\left[r\left(t+m\right)\right]$ are defined as $\sum_{n=1}^{3}\sigma_n\cdot g_n\left(\phi,m\right)\cdot dW_n\left(t\right)$, where σ_n are constant standard deviation parameters and $dW_n\left(t\right)$ are independent Wiener variables under the physical (i.e not risk-neutral) measure. The associated forward rate curve derived via the Heath, Jarrow and Morton (1992) (hereafter HJM) framework is then:

$$f\left(t,m\right)=\sigma_1\rho_1 m+\sum_{n=1}^{3}\beta_n\left(t\right)\cdot g_n\left(\phi,m\right)-\sum_{n=1}^{3}\sigma_n^2\cdot h_n\left(\phi,m\right) \tag{3}$$

where $\beta_n\left(t\right)=\gamma_n+\lambda_n\left(t\right)$, γ_n are constant term premia parameters derived as $\gamma_1=\frac{1}{\phi}(-\sigma_2\rho_2+\sigma_3\rho_3)$, $\gamma_2=\frac{1}{\phi}\left(-\sigma_2\rho_2-2\sigma_3\rho_3\right)$, $\gamma_3=\frac{1}{\phi}\sigma_3\rho_3$, ρ_n are constant market prices of risk, and $h_n(\phi,m)$ are time-invariant functions of maturity derived as $h_1(\phi,m)=\frac{1}{2}m^2$, $h_2(\phi,m)=\frac{1}{2\phi^2}\left[1-\exp\left(-\phi m\right)\right]^2$, $h_3(\phi,m)=\frac{1}{2\phi^2}[1-\exp\left(-\phi m\right)-2m\phi\exp\left(-\phi m\right)]^2$.

Hence, the ANS model from Krippner (2006) retains the essence of the NS approach, in that the yield curve at time t is still represented with just three coefficients $\beta_1\left(t\right)$, $\beta_2\left(t\right)$, and $\beta_3\left(t\right)$. However, the subtle enhancement to NS models is that the ANS model embeds consistency between the dynamics of the time series of ANS coefficients and the effect those dynamics have on each observation of the yield curve under the standard assumption that the market will price securities to exclude arbitrage opportunities. The estimation of the ANS model also retains the essence of the NS approach, in that the ANS coefficients at time t are estimated from the yield curve data at time t, and the two free parameters ϕ and ρ_1 are determined to provide an appropriate fit to the overall series of historical yield curve data.

Anticipating the complete discussion of the data and the empirical application from section 5, figure 2 illustrates an example of the results from estimating the ANS model by fitting the yield curve data for September 2003. Figure 3 illustrates the time series of a representative yield and slope measure from the time series of yield curve data. Figure 4 plots the time series of the estimated ANS coefficients $\beta_1\left(t\right)$, $\beta_2\left(t\right)$, and $\beta_3\left(t\right)$ obtained from the full sample of yield curve data.

[Figure 2 here], [Figure 3 here], [Figure 4 here]

3 An economic foundation for the ANS model of the yield curve

This section proceeds in three sub-sections to establish a macroeconomic foundation for the ANS model of the yield curve. Section 3.1 specifies a generic model of the economy and notes how the factors in that economy may be aggregated into the macroeconomic quantities of output growth and inflation. Section 3.2 derives the yield curve associated with the generic economy under rational expectations and shows that the level and shape of that yield curve is composed of expectations of output growth and inflation. Section 3.3 then explicitly relates the derived yield curve model and its macroeconomic foundation to the ANS model thereby allowing the state variables and parameters of ANS model to be interpreted in terms of the state variables and parameters of the original model of the economy.

Note that in keeping with the minimal specification of the ANS model, and also to maximise the transparency of the concepts and derivations, the economic model is developed using independent factors with uncorrelated Gaussian dynamics. However, with the usual trade-off against parsimony and transparency, the assumptions could be relaxed and the model generalised if particular applications required more sophistication.[5]

3.1 A generic model for the economy

This model outlined in this section is a generalised multifactor version of the standard continuous-time general-equilibrium economy proposed by Cox et al. (1985*a*) (hereafter CIR), specified with Vasicek (1977) dynamics (i.e Gaussian innovations). Hereafter, this is abbreviated to the generalised CIR-Vasicek model, or the GCV model for short. Note that the GCV model has some parallels with the model of Berardi and Esposito (1999) (hereafter BE) that is also based on a multifactor CIR economy embedding Vasicek dynamics (although with constant steady-state parameters and a single inflation state variable), and so references to the BE model are made as appropriate.

The GCV economy is based on an arbitrary number J real factors of production (e.g capital, labour, productivity, etc., all potentially by industry sector etc.), each with its own associated deflator/inflation factor. The dynamics of the GCV economy are represented by $2J$ processes analogous to the Vasicek (1977) specification, i.e:

$$ds_j(t) = -\kappa_j [s_j(t) - \theta_j(t)] \, dt + \sigma_{1,j} dz_{1,j}(t) \tag{4}$$

where $s_j(t)$ for $j = 1$ to J are the real state variables representing instantaneous growth on returns to the factors of production in the economy at time t; κ_j are positive constant mean-reversion parameters; $\theta_j(t)$ are the steady-state (i.e long-run) values of $s_j(t)$ that are allowed to vary stochastically over time as $d\theta_j(t) = \sigma_{0,j} dz_{0,j}(t)$; $\sigma_{0,j}$ and $\sigma_{1,j}$ are positive constant standard deviation parameters with $\sigma_{0,j} \ll \sigma_{1,j}$; and $dz_{0,j}(t)$ are $dz_{1,j}(t)$ are independent Wiener variables under the physical measure. For $j = J+1$ to $2J$, $s_j(t)$ are the inflation state variables. As noted in BE p. 155, these have the form $s_{J+j}(t) = \pi_j(t) - \sigma_{p,j}^2$, where $\pi_j(t)$ is the instantaneous rate of inflation for the factor of production j and $\sigma_{p,j}^2$ is a constant parameter representing the variance of instantaneous changes in the deflator j. Similarly, $\theta_{J+j}(t) = \theta_{\pi,j}(t) - \sigma_{p,j}^2$, where $\theta_{\pi,j}(t)$ is the steady-state rate of inflation for the factor of production j. The remaining parameters for the inflation state variables are analogous to the real state variables.

The state variables and steady-state variables of the GCV model may be accumulated into four macroeconomic quantities. Respectively, with brief justification:

1. $dY(t) = \sum_{j=1}^{J} s_j(t)$ is (instantaneous) real output growth. As noted in BE p. 147, the real wealth of individuals in a CIR economy is completely invested in the production process, the return on that production process is the income of individuals, and individuals optimally allocate that income to consumption and investment. Hence, the sum of growth on the returns to the factors of production in the economy is production growth, which equals income growth and expenditure growth, and these are the three standard expressions of output growth for the economy.

2. $dY^*(t) = \sum_{j=1}^{J} \theta_j(t)$ is (instantaneous) real steady-state output growth. If the real returns to the factors of production are all growing at their steady-state values, then real output must be growing at its steady-state value.

[5] In particular, without changing the nature of the framework in this article, it is straightforward to allow for interdependence between the factors and for correlated innovations by applying principal components (a proof is available from the author on request). The square-root innovations of Cox, Ingersoll and Ross (1985*b*), weakly mean-reverting steady-state variables, and time-varying volatilities and market prices of risk could be readily accomodated, but would also require corresponding changes to the ANS model. In principle, more complex aspects such as Phillips curve inflation/output relationships, Taylor rule monetary policy reaction functions, monetary policy credibility effects, etc., could be incorporated by specifying them within the economic model, deriving the associated forward rate curve, and adding the required components to the ANS model.

3. $dP(t) = \sigma_P^2 + \sum_{j=J+1}^{2J} s_j(t)$ is (economy-wide instantaneous) inflation. This follows as for item 1, but substituting the inflation component of nominal wealth and abbreviating $\sum_{j=J+1}^{2J} \sigma_{p,j}^2$ to σ_P^2, which is the variance of inflation.

4. $dP^*(t) = \sigma_P^2 + \sum_{j=J+1}^{2J} \theta_j(t)$ is (instantaneous) steady-state inflation. This follows as for item 2, but substituting the steady-state inflation variables and using σ_P^2 from item 3.

The practical intuition of the GCV macroeconomy is therefore an anticipated equilibrium composed of expected long-term potential output growth $dY^*(t)$ plus expected long-term inflation $dP^*(t)$. Regarding dynamics, $dY^*(t)$ and $dP^*(t)$ evolve over time as low-variance Gaussian random walks, respectively $\sum_{j=1}^{J} \sigma_{0,j} dz_{0,j}(t)$ and $\sum_{j=J+1}^{2J} \sigma_{0,j} dz_{0,j}(t)$. These innovations allow for persistent "shocks" to $dY^*(t)$ and $dP^*(t)$, which in turn relate back to persistent variations in the underlying factors of production (e.g changes to anticipated long-term population growth, or changes to expected productivity growth due to technological change, etc.). Current output growth $dY(t)$ and inflation $dP(t)$ can vary from their steady-state values, but are always expected to return to $dY^*(t)$ and $dP^*(t)$ over time due to the mean-reverting dynamics of equation 4. Regarding the stochastic components, $dY(t)$ and $dP(t)$ will be impacted respectively by the innovations $\sum_{j=1}^{J} \sigma_{1,j} dz_{1,j}(t)$ and $\sum_{j=J+1}^{2J} \sigma_{1,j} dz_{1,j}(t)$, which allow for transitory "shocks" to output growth and inflation (e.g credit channel restrictions on capital investment, terms of trade shifts, etc.).

3.2 The yield curve associated with the GCV economy

As noted in BE, the nominal short rate at any given time is the sum of the state variables $s_j(t)$; i.e $r(t) = \sum_{j=1}^{2J} s_j(t)$. That defines the short rate and its dynamics for the GCV economy, and the associated forward rate curve may then be derived via the HJM framework. The full details of these calculations are contained in appendix B, with just the relevant results summarised below in two propositions, each followed by a brief discussion of their practical intuition.

Proposition 1 ***The GCV expected path of the short rate***

The expected path of the short rate for the economy specified in equation 4 is:

$$
\begin{aligned}
E_t[r(t+m)] &= \sum_{j=1}^{2J} E_t[s_j(t+m)] && (5)\\
&= \sum_{j=1}^{2J} \theta_j(t) + \sum_{j=1}^{2J} [s_j(t) - \theta_j(t)] \cdot \exp(-\kappa_j m) && (6)\\
&= dY^*(t) + dP^*(t) - \sigma_P^2 + E_t[dX(t+m)] && (7)
\end{aligned}
$$

where $E_t[s_j(t+m)]$ is the expected path of the state variable j as at time t and as a function of future horizon m; $E_t[dX(t+m)]$ is the expected path of the deviation of output growth plus inflation from their steady-state values, as at time t and as a function of future horizon m. Specifically, $E_t[dX(t+m)] = E_t[dY(t+m) + dP(t+m) - dY^*(t+m) - dP^*(t+m)]$ where each of the latter are the values of the macroeconomic quantities defined in section 3.1 as at time t and as a function of future horizon m.

Proof. In appendix B.2 ■

The intuition underlying equation 7 is essentially that the current value of the short rate $r(t)$ equates to current output growth $dY(t)$ plus inflation $dP(t)$ less inflation variance σ_P^2, the long-horizon expected value of the short rate equates to long-term potential output growth $dY^*(t)$ plus expected long-term inflation $dP^*(t)$ less inflation variance σ_P^2,[6] and for intermediate horizons the

[6] Formally, $\lim_{m\to 0} E_t[r(t+m)] = r(t) = \sum_{j=1}^{2J} s_j(t) = dY(t) + dP(t) - \sigma_P^2$, and $\lim_{m\to\infty} E_t[r(t+m)] = \sum_{j=1}^{2J} \sum_{j=1}^{2J} \theta_j(t) = dY^*(t) + dP^*(t) - \sigma_P^2$.

expected path of the short rate reflects the expected mean reversion of output growth plus inflation back to their steady-state values.

Proposition 2 ***The GCV forward rate curve***

The (default-free) forward rate curve for the economy specified in equation 4 is:

$$
\begin{aligned}
f(t,m) &= \sum_{j=1}^{2J} \theta_j(t) + m \cdot \sum_{j=1}^{2J} \sigma_{0,j}\rho_{0,j} - m^2 \cdot \sum_{j=1}^{2J} \frac{1}{2}\sigma_{0,j}^2 \\
&\quad + \sum_{j=1}^{2J} [s_j(t) - \theta_j(t)] \cdot \exp(-\kappa_j m) + \sum_{j=1}^{2J} \left[\sigma_{1,j}\rho_{1,j} - \sigma_{0,j}\rho_{0,j}\right] \cdot B_j(m) \qquad (8) \\
&\quad - \sum_{j=1}^{2J} \frac{1}{2}\left[\sigma_{1,j}^2 - \sigma_{0,j}^2\right] \cdot [B_j(m)]^2 \\
&= dY^*(t) + dP^*(t) - \sigma_P^2 + E_t[dX(t+m)] + TP(m) + VE(m) \qquad (9)
\end{aligned}
$$

where $B_j(m) = \frac{1}{\kappa_j}[1 - \exp(-\kappa_j m)]$ is the Vasicek (1977) functional form, $\rho_{0,j}$ and $\rho_{1,j}$ are respectively constant market prices of risk associated with the innovations $dz_{0,j}(t)$ and $dz_{1,j}(t)$, $TP(m)$ is the term premium as a function of time-to-maturity m that collects the terms $m \cdot \sum_{j=1}^{2J} \sigma_{0,j}\rho_{0,j} + \sum_{j=1}^{2J}\left[\sigma_{1,j}\rho_{1,j} - \sigma_{0,j}\rho_{0,j}\right] \cdot B_j(m)$, and $VE(m)$ is the volatility effect as function of time-to-maturity m that collects the terms $-m^2 \cdot \sum_{j=1}^{2J} \frac{1}{2}\sigma_{0,j}^2 - \sum_{j=1}^{2J} \frac{1}{2}\left[\sigma_{1,j}^2 - \sigma_{0,j}^2\right] \cdot [B_j(m)]^2$.

Proof. In appendix B.3 ■

The GCV forward rate curve is the expected path of the short rate with two series of adjustments for the effects of dynamics so that arbitrage opportunities are precluded. One series of adjustments is for the pure effects that volatilities (i.e $\sigma_{0,j}$ and $\sigma_{1,j}$) have on expected returns relative to a compounding rolling investment in the short rate. The other series of adjustments is for the combined effects of volatilities (i.e quantities of risk) and the market prices of risk (i.e $\rho_{0,j}$ and $\rho_{1,j}$). This adjustment represents a term premium, which compensates investors for bearing the risk of investing in fixed interest securities of time-to-maturity m relative to a risk-free compounding rolling investment in the short rate over time t to $t+m$. Note that this relative risk exists even with default-free securities; i.e changes to $f(t,m)$ via the innovations $\sigma_{0,j} dz_{0,j}(t)$ and $\sigma_{1,j} dz_{1,j}(t)$ will change the interest rate curve $R(t,m) = \frac{1}{m}\int_0^m f(t,m)dm$ and the discount function $\exp[-m \cdot R(t,m)]$, and will therefore lead to unanticipated changes in the prices of the fixed interest securities that compose the yield curve.

3.3 Relating the ANS model to the yield curve of the generic economy

An explicit macroeconomic foundation for the ANS model from section 2 may now be provided by comparing it to the functional form and dynamics for the GCV expected path of the short rate and the forward rate curve that have been defined in terms of economic state variables and macroeconomic quantities. Proposition 4 summarises the results of that comparison for the Level component of the ANS model, Proposition 5 summarises the results for the Slope and Bow components, and both are followed by their respective proofs and a discussion of their practical intuition.

Proposition 3 ***The ANS Level component***

The key relationship between the ANS Level component and the GCV steady-state components is:

$$\beta_1(t) - \gamma_1 = dY^*(t) + dP^*(t) - \sigma_P^2 \qquad (10)$$

For completeness, the dynamics of the ANS Level component relate to the dynamics of the GCV model as follows: $\sigma_1 dW_1(t) = \sum_{j=1}^{2J} \sigma_{0,j} dz_{0,j}(t)$, $\sigma_1^2 = \sum_{j=1}^{2J} \sigma_{0,j}^2$, and $\sigma_1\rho_1 = \sum_{j=1}^{2J} \sigma_{0,j}\rho_{0,j}$.

Proof. The only component that is constant by future horizon in equation 2 is $\lambda_1(t) \cdot g_1(\phi, m) = \lambda_1(t) = \beta_1(t) - \gamma_1$. Equating the latter to the components that are constant by future horizon in equations 6 and 7 gives $\beta_1(t) - \gamma_1 = \sum_{j=1}^{2J} \theta_j(t) = dP^*(t) + dY^*(t) - \sigma_P^2$. Regarding dynamics: equating the stochastic components of $\beta_1(t)$ and $dP^*(t) + dY^*(t)$ gives $\sigma_1 dW_1(t) = \sum_{j=1}^{2J} \sigma_{0,j} dz_{0,j}(t)$; equating the components of m^2 from equations 3 and 8 gives $\sigma_1^2 \cdot h_1(\phi, m) = \sigma_1^2 \cdot \frac{1}{2}m^2 = m^2 \cdot \sum_{j=1}^{2J} \frac{1}{2}\sigma_{0,j}^2$, and so $\sigma_1^2 = \sum_{j=1}^{2J} \sigma_{0,j}^2$; and equating the components of m from equations 3 and 8 gives $\sigma_1 \rho_1 m = m \cdot \sum_{j=1}^{2J} \sigma_{0,j}\rho_{0,j}$, and so $\sigma_1 \rho_1 = \sum_{j=1}^{2J} \sigma_{0,j}\rho_{0,j}$. ■

Hence, the Level coefficient from the ANS model equates to within a parametric term premium to steady-state output growth plus steady-state inflation less inflation variance within the GCV economy. Moreover, the entire Level component of the ANS forward rate curve including the effects of dynamics, i.e $\sigma_1\rho_1 m + \beta_n(t) - \sigma_1^2 \cdot \frac{1}{2}m^2$, equates to the entire steady-state component of the GCV forward rate equation, i.e the first line of equation 8. The latter correspondence carries through to the interest rate curves and discount functions for the ANS and GCV models, and so ensures that when yield curve data for default-free securities observed at time t are fitted using the ANS model, the Level coefficient $\beta_1(t)$ will be a consistent estimate to within the term premium γ_1 of the sum of the steady-state components of the GCV model as at time t.

Proposition 4 ***The ANS Slope and Bow components***

The key relationship between the ANS Slope and Bow components and the non-steady-state GCV components is:

$$-\sum_{n=2}^{3} [\beta_n(t) - \gamma_n] \cdot g_n(\phi, m) \simeq E_t[dX(t+m)] \tag{11}$$

For completeness, the dynamics of the ANS Slope and Bow components relate to the GCV model as follows: $\sum_{n=2}^{3} \sigma_n \cdot g_n(\phi, m) \cdot dW_n(t) \simeq -\sum_{j=1}^{2J} [\sigma_{1,j} dz_{1,j}(t) - \sigma_{0,j} dz_{0,j}(t)] \cdot \exp(-\kappa_j m)$, $\sum_{n=2}^{3} \sigma_n^2 \cdot h_2(\phi, m) \simeq -\sum_{j=1}^{2J} \frac{1}{2}\left[\sigma_{1,j}^2 - \sigma_{0,j}^2\right] \cdot [B_j(m)]^2$, *and* $\sum_{n=2}^{3} \gamma_n \cdot g_n(\phi, m) \simeq \sum_{j=1}^{2J} \left[\sigma_{1,j}\rho_{1,j} - \sigma_{0,j}\rho_{0,j}\right] \cdot B_j(m)$.

Proof. Equating the non-steady-state components of equations 6 and 7, $E_t[dX(t+m)] = \sum_{j=1}^{2J} [s_j(t) - \theta_j(t)] \cdot \exp(-\kappa_j m)$. Define $\phi = \text{median}(\kappa_j)$, which is a positive constant because all κ_j are positive constants. Expressing $\kappa_j = \phi(1 + \Delta_j)$, where Δ_j is the relative difference of κ_j from ϕ, $\sum_{j=1}^{2J} [s_j(t) - \theta_j(t)] \cdot \exp(-\kappa_j m) = \exp(-\phi m) \cdot \sum_{j=1}^{2J} [s_j(t) - \theta_j(t)] \cdot \exp(-\Delta_j \phi m)$. The first-order Taylor expansion of $\exp(-\Delta_j \phi m)$ around $\Delta_j = 0$ is $\exp(-\Delta_j \phi m) \simeq 1 - \Delta_j \phi m$, and so $E_t[dX(t+m)] \simeq \exp(-\phi m) \cdot \left[\sum_{j=1}^{2J} [s_j(t) - \theta_j(t)] - \phi m \cdot \sum_{j=1}^{2J} [s_j(t) - \theta_j(t)] \Delta_j\right]$. The latter may be expressed using the functions of future horizon m for the ANS expected path of the short rate from equation 2; i.e $E_t[dX(t+m)] \simeq \left[-\sum_{j=1}^{2J} [s_j(t) - \theta_j(t)] \cdot \left(1 - \frac{1}{2}\Delta_j\right)\right] \cdot [-\exp(-\phi m)] + \left[-\sum_{j=1}^{2J} [s_j(t) - \theta_j(t)] \frac{1}{2}\Delta_j\right] \cdot [-\exp(-\phi m)(-2\phi m + 1)] = \lambda_2(t) \cdot g_2(\phi, m) + \lambda_3(t) \cdot g_3(\phi, m) = [\beta_2(t) - \gamma_2] \cdot [-\exp(-\phi m)] + [\beta_3(t) - \gamma_3] \cdot [-\exp(-\phi m)(-2\phi m + 1)]$. Hence, $\beta_2(t) - \gamma_2 = -\sum_{j=1}^{2J} [s_j(t) - \theta_j(t)] \cdot \left(1 + \frac{1}{2}\Delta_j\right)$ and $\beta_3(t) - \gamma_3 = -\sum_{j=1}^{2J} [s_j(t) - \theta_j(t)] \frac{1}{2}\Delta_j$. Regarding dynamics, the stochastic components of the GCV model are $-\sum_{j=1}^{2J} [\sigma_{1,j} dz_{1,j}(t) - \sigma_{0,j} dz_{0,j}(t)] \cdot \exp(-\kappa_j m)$, which may be approximated as $\sum_{n=2}^{3} \sigma_n \cdot g_n(\phi, m) \cdot dW_n(t)$ by following the first-order Taylor expansion approach outlined above. Finally, Krippner (2006) has already shown that the components $\sum_{n=2}^{3} \lambda_n(t) \cdot g_n(\phi, m)$ and $\sum_{n=2}^{3} \sigma_n \cdot g_n(\phi, m) \cdot dW_n(t)$ within the HJM framework produce a term premium component $\sum_{n=2}^{3} \gamma_n \cdot g_n(\phi, m)$ and a volatility effect $\sum_{n=2}^{3} \sigma_n^2 \cdot h_2(\phi, m)$. Matching these respectively to their non-steady-state GCV counterparts gives $\sum_{n=2}^{3} \gamma_n \cdot g_n(\phi, m) \simeq \sum_{j=1}^{2J} \left[\sigma_{1,j}\rho_{1,j} - \sigma_{0,j}\rho_{0,j}\right] \cdot B_j(m)$ and $\sum_{n=2}^{3} \sigma_n^2 \cdot h_2(\phi, m) \simeq -\sum_{j=1}^{2J} \frac{1}{2}\left[\sigma_{1,j}^2 - \sigma_{0,j}^2\right] \cdot [B_j(m)]^2$. ■

The first level of intuition on the result above is that, because the Level components of the ANS model equate to the steady-state components of the GCV model, the "remainder" of the default-free yield curve as estimated by the ANS model (i.e the Slope plus Bow components plus the yield residuals) must therefore reflect the non-steady-state components of the GCV model relative to their steady-state components. More precisely, the proof for Proposition 4 shows that the combination of the ANS Slope plus Bow components equates to within a parametric term premium function to the first-order Taylor approximation around the median value of the mean reversions κ_j of the $2J$ state variables relative to their steady-state values within the GCV economy. Hence, the ANS yield residuals correspond to the second-order and higher terms from the Taylor expansion of the GCV model rate, which may be assumed to negligible given the close empirical fit of the ANS model (and NS models generally) to observed yield curve data.[7]

In summary then, using just three state variables $\beta_1(t)$, $\beta_2(t)$, and $\beta_3(t)$, and two free parameters ϕ and ρ_1, the ANS forward rate curve provides a practically-tenable representation of the generic GCV forward rate curve containing $2J$ state variables with associated standard deviations and market prices of risk, $2J$ steady-state variables with associated standard deviations and market prices of risk, $2J$ mean-reversion parameters, and J inflation variance parameters.[8] The ANS model therefore provides a reduction in dimensionality that is commonly undertaken using latent factor models for the yield curve, such as in Ang and Piazzesi (2003) and in the Diebold et al. (2006) application of the NS model, but with a theoretical justification for the ANS components via the precise first-order approximation to the GCV model. The aggregation of the GCV state variables and steady-state variables into four macroeconomic quantities (i.e steady-state output growth, steady-state inflation, output growth, and inflation) then provides the macroeconomic foundation for the ANS model.

4 Econometric relationships for the ANS model coefficients, and inflation and output growth

The ANS framework (i.e the ANS model from section 2 with its macroeconomic foundation from section 3) has an obvious connection to the field of macro-finance, given that it offers a theoretically-rigorous yet parsimonious and intuitive connection between the yield curve, output growth, and inflation. As a simple and practical illustration of applying it as a macro-finance tool, the ANS framework is used in this section to derive theoretical econometric relationships between the yield curve, output growth, and inflation. This essentially requires converting the ANS framework relationships in continuous time to discrete-time, and an annual basis is used to make for ready comparison to the existing empirical macro-finance literature.[9]

The elements of equation 10 are all contemporaneous, and so its discrete-time version may simply be written as:

$$\beta_{1,t} - [\Delta Y^*(t) + \Delta P^*(t)] = \alpha^* + \varepsilon_t^* \tag{12}$$

where $\beta_{1,t}$ is the annual average of the estimated Level coefficients over the prior year to time t, $\Delta Y^*(t)$ is annual steady-state output growth to time t, $\Delta P^*(t)$ is annual steady-state inflation to

[7] This implies that $|\Delta_j| < 1$. Note also that the ANS model could be extended arbitrarily by adding higher-order exponential-polynomial functions, which would correspond to additional terms in the Taylor approximation of the non-steady state components of the GCV model. In this sense, the approximation of the GCV model with the ANS model is "natural". Conversely, arbitrary approximations based on other functions, e.g simple polynomials as in McCulloch (1971) or Chebyshev polynomials as in Pham (1998), are "unnatural" because the addition of each higher-order term does not directly correspond to an extra term in the Taylor expansion.

[8] For example, even a minimal CGV model based on a single industry with a single factor of production (i.e $J = 1$) would require two state variables (output and inflation), two steady-state variables, and 11 parameters. A more sophisticated specification based on just two industries and the three typical factors of production (i.e capital, labour, and total factor productivity) would give six factors of production (i.e $J = 6$), and would require 12 state variables, 12 steady-state variables, and 66 parameters.

[9] Of course, the ANS framework would be applicable to any other time-step and horizon (and to forward horizons) by straightforward modifications to the derivation in this section.

time t,[10] α^* captures the parameters γ_1 and σ_P^2 and any other systematic differences between $\beta_{1,t}$ and $[\Delta Y^*(t) - \Delta P^*(t)]$ (e.g persistent measurement biases in the macroeconomic data), and ε_t^* captures any time-varying components (e.g subsequent revisions to macroeconomic data). Assuming ε_t^* to be stationary, equation 12 represents a (1,-1) cointegrating relationship between $\beta_{1,t}$ and $[\Delta Y^*(t) - \Delta P^*(t)]$. Note that all of the data in equation 12 are Gaussian processes, and so OLS estimation and standard unit root tests are applicable.

Equation 11 is an intertemporal relationship between the estimated Slope and Bow coefficients at time t and $E_t[dX(t+m)]$, the expected path of the deviation of output growth plus inflation from their steady-state values over time t to $t+m$. The discrete-time measure for the latter on an annual basis (i.e future horizon $m=1$) is $E_t[\Delta X(t+1)] = E_t[\Delta Y(t+1) + \Delta P(t+1) - \Delta Y^*(t+1) - \Delta P^*(t+1)]$, where the quantity on the right-hand side is the expected annual change of $dY(t)$ and $dP(t)$ less $dY^*(t)$ and $dP^*(t)$. The corresponding quantity for the left-hand side of equation 11 may be calculated directly for arbitrary m as $-\frac{1}{m}\int_t^{t+m}\left[\sum_{n=2}^{3}[\beta_n(t)-\gamma_n]\cdot g_n(\phi,m)\right]dm = [\beta_n(t)-\gamma_n]\cdot\sum_{n=2}^{3} q_n(m)$, where $q_n(m) = -\frac{1}{m}\int_0^m g_n(\phi,m)dm$, and the two required integrals are $q_2(m) = -\frac{1}{\phi m}[\exp(-\phi m)-1]$, and $q_3(m) = \frac{1}{\phi m}[2\phi m\exp(-\phi m)+\exp(-\phi m)-1]$. For the year-ahead horizon (i.e $m=1$) in this application and using $\phi = 1.03$ as obtained in the empirical application of the following section, $q_2(1) = 0.6231$ and $q_3(1) = 0.0878$. Hence, the resulting relationship between $E_t[\Delta X(t+1)]$ and the estimated Slope and Bow coefficients at time t is:

$$E_t[\Delta X(t+1)] = \alpha_{0,1} + \alpha_{1,1}\cdot[\beta_2(t)\cdot q_2(1) + \beta_3(t)\cdot q_3(1)] + \varepsilon(t+1) \tag{13}$$

where $\alpha_{0,1}$ captures the parameters $[\gamma_2\cdot q_2(1) + \gamma_3\cdot q_3(1)]$ and any other systematic effects, and $\varepsilon(t+1)$ captures any time-varying effects. All of the data in equation 13 are Gaussian processes, and so OLS estimation is applicable.

There are close parallels between the econometric relationships derived above and the OLS regressions and Diebold et al. (2006) VAR that have been applied within the existing empirical macro-finance literature. Firstly, the cointegrating relationship in equation 12 is the same form as typical tests for (1,-1) cointegration between term interest rates and inflation as motivated by the Fisher hypothesis. Equation 12 also embeds the two-way relationship established in Diebold et al. (2006) between inflation and the NS Level coefficient $\beta_1^{\mathrm{NS}}(t)$, assuming (reasonably) that any effects of not consistently accounting for dynamics in $\beta_1^{\mathrm{NS}}(t)$ are relatively minor compared to the typical variation in $\beta_1(t)$ over time. Secondly, the relationship in equation 13 is the same form as typical intertemporal OLS regressions of current annual output growth or current annual inflation on the lagged slope of the yield curve; i.e $\Delta Y(t)$ or $\Delta P(t) = \alpha_0 + \alpha_1\cdot[\mathrm{GS10}(t-L)\text{-}\mathrm{TB3}(t-L)] + \varepsilon_t$, where L represents the lag length (typically four quarters). Equation 13 also embeds the two-way relationships established in Diebold et al. (2006) between output growth or inflation and the NS Slope coefficient, again assuming $\beta_2^{\mathrm{NS}}(t)$ moves similarly to $\beta_2(t)$ over time.

Importantly, however, the ANS framework adds a succinct theoretical basis for why the data should support those empirically-established relationships. Regarding cointegration, inflation and the long-maturity level of the yield curve (as captured by term interest rates or the $\beta_1^{\mathrm{NS}}(t)$ coefficient) both reflect the process for expected long-term inflation. Regarding the intertemporal regressions, future output growth (or future inflation) and the shape of the yield curve (as captured by $[\mathrm{GS10}(t)\text{-}\mathrm{TB3}(t)]$ or the $\beta_2^{\mathrm{NS}}(t)$ coefficient) both reflect the process for expected short-term output growth (or expected short-term inflation). The succinct theoretical basis provided by the ANS framework also highlights how the empirical relationships may be undermined (aside from

[10] Formally and for arbitrary m, $\frac{1}{m}\int_{t-m}^{t}\beta_1(u)\,du$ corresponds to $\frac{1}{m}\int_{t-m}^{t}[dY^*(u)+dP^*(u)]\,du$. For $m=1$, $\int_{t-1}^{t}\beta_1(u)\,du = \beta_{1,t}$ and $\int_{t-1}^{t}[dY^*(u)+dP^*(u)]\,du = [\Delta Y^*(t)+\Delta P^*(t)]$.

measurement errors), i.e: (1) when expectations of output growth and inflation are not efficiently incorporated into the yield curve; (2) when those expectations prove inaccurate (i.e if "shocks" to output growth and/or inflation are relatively large); and (3) when changes to term premia and volatilities become a material consideration.

Finally, the econometric relationships derived via the ANS framework suggest several extensions to the relationships estimated in the existing empirical macro-finance literature. Firstly, decomposing future output growth and inflation data into short-term and long-term components should improve the empirical correlations with yield curve data. Secondly, yield curve data should relate better to economy-wide measures of inflation, rather than to consumer price inflation measures. Thirdly, the VAR application of Diebold et al. (2006) finds no statistically-significant relationship between inflation or output growth and the Bow (or Curvature) coefficient $\beta_3^{\mathrm{NS}}(t)$ in its own right, but the ANS framework suggests that $\beta_3^{\mathrm{NS}}(t)$ should play a role in relating the shape of the yield curve to future inflation and output growth.

5 An empirical application to US data

This section estimates the econometric relationships derived from the ANS model framework using US data. To make the results directly comparable to the existing empirical macro-finance literature, the analysis is undertaken in-sample using annual data at a quarterly frequency. Section 5.1 outlines that data, and section 5.2 discusses the results of the estimations.

5.1 Description of the yield curve, output growth, and inflation data

The interest rate data used in the empirical application are the monthly averages of the federal funds rate, the 3-month Treasury bill rate, and the 1-year, 3-year, 5-year, 10-year, and 20-year or 30-year constant-maturity bond rates,[11] all obtained from the online Federal Reserve Economic Database (hereafter the FRED) on the Federal Reserve Bank of St. Louis website. The sample period is July 1954 (the first month federal funds rate data is available) to February 2008 (the last month available at the time of the analysis), giving 644 monthly observations of the yield curve.

The monthly time series of ANS Level, Slope, and Bow coefficients derived from the yield curve data have already been illustrated in figure 4, and these were calculated following the method outlined in appendix C of Krippner (2006). To briefly summarise, the federal funds rate and the 3-month Treasury bill rate have defined zero-coupon cashflows (i.e settlement price with principle plus interest at maturity), and the government bond cashflows assume that the yield-to-maturity corresponds to a par semi-annual bond (settlement price of 1, six-monthly coupons of half the yield-to-maturity, and principle of 1 plus coupon at maturity). The estimation of $\beta_1(t)$, $\beta_2(t)$, and $\beta_3(t)$ for each yield curve observation is undertaken via the least squares minimisation of discounted cashflows with given values of ϕ and ρ_1, and estimates of σ_1, σ_2, and σ_3 obtained as the annualised standard deviations of changes in the coefficients $\beta_1(t)$, $\beta_2(t)$, and $\beta_3(t)$ from an initial pass of the entire historical yield curve data series with all $\sigma_n = 0$. A grid search is used to determine that the point estimates of the free parameters $\phi = 1.03$ and $\rho_1 = 2.66\%$ provide the best overall fit to the entire historical yield curve data series, and the associated point estimates of the annualised standard deviations are $\sigma_1 = 0.77\%$, $\sigma_2 = 2.26\%$, and $\sigma_3 = 1.74\%$.

The data series $\beta_{1,t}$ is then calculated as the twelve-month trailing average of $\beta_1(t)$, and taking the last month of each quarter provides the relevant quarterly data (214 observations) for $\beta_{1,t}$, $\beta_2(t)$ and $\beta_3(t)$ in equations 12 and 13.

Direct measurements of the idealised GCV macroeconomic quantities in equations 12 and 13 are not available over the full time period,[12] and so proxy data are necessarily required. Hence, the

[11] 20-year data is unavailable from January 1987 to September 1993, and so 30-year data (with a 30-year maturity) is used during this period for the estimation of the ANS model.

[12] For example, the Survey of Professional Forecasters data from the Philidelphia Federal Reserve website only contains long-term (i.e 10-year) expectations of inflation and output growth from 1991.

proxy for steady-state output growth $\Delta Y^*(t)$ is the annual change in the Congressional Budget Office potential GDP (see Congressional Budget Office (2001) for calculation details), and the proxy for steady-state inflation $\Delta P^*(t)$ is the annual change in the GDP deflator.

Following the existing empirical literature, e.g see Estrella et al. (2003), equation 13 may be estimated using realised values of $\Delta X(t)$ regressed against the values of $[\beta_2(t) \cdot q_2(1) + \beta_3(t) \cdot q_3(1)]$ lagged four quarters. From the previous section, $\Delta X(t) = \Delta Y(t) + \Delta P(t) - \Delta Y^*(t) - \Delta P^*(t)$, and the proxies for $\Delta Y^*(t)$ and $\Delta P^*(t)$ have already been discussed. Output growth $\Delta Y(t)$ is proxied by the annual change in GDP, and inflation $\Delta P(t)$ is proxied by the annual change in the GDP deflator. Note that because $\Delta P^*(t)$ and $\Delta P(t)$ have been proxied by the same quantity, $\Delta P^*(t) - \Delta P(t) = 0$, and so $\Delta X(t) = \Delta Y(t) - \Delta Y^*(t)$ for the empirical analysis in this article.

The quarterly index levels for the macroeconomic data series mentioned are available from the FRED, and annual changes are calculated from the logarithms of those levels. To allow a visual inspection of some of the relationships to be estimated, figure 5 plots the time series of $\beta_{1,t}$ and $[\Delta Y^*(t) + \Delta P^*(t)]$ data, and figure 6 plots the difference between those series. Figure 7 illustrates the time series of $\Delta Y^*(t)$ and $\Delta Y(t)$ that are used to calculate $\Delta X(t)$, and figure 8 plots $\Delta X(t)$ and $[\beta_2(t) \cdot q_2(1) + \beta_3(t) \cdot q_3(1)]$ lagged four quarters.

[Figure 5 here] [Figure 6 here], [Figure 7 here], [Figure 8 here]

As a further comparison to the existing empirical macro-finance literature, estimations analogous to equations 12 and 13 are also undertaken based on annual GDP growth and annual changes in the GDP deflator individually, and also using the original interest rate data rather than the ANS coefficients. Specifically, the additional interest data are the trailing annual average of the monthly 3-year government bond yield GS3(t) illustrated in figure 3 (hereafter denoted as AGS3_t), and the slope of the yield curve $[\text{GS10}(t)\text{-TB3}(t)]$ from figure 3.

5.2 Empirical results and discussion

Table 1 contains the results of standard augmented Dickey-Fuller (ADF) and Phillips-Perron (PP) unit root tests with an estimated constant for all of the data series used in the empirical analysis. The lag lengths for the ADF tests are the theoretical three quarters (given that annual data at a quarterly frequency induces MA(3) autocorrelation) and an automatic number of lags selected using the method from Hamilton (1994) p. 530, which resulted in lag lengths ranging from 0 to 26 quarters. Similarly, the Newey and West (1987) window widths for the PP tests are three quarters and an automatic window width selected using the method from Newey and West (1994), which resulted in window widths ranging from 7 to 11 quarters. Unit root tests for first differences are also undertaken as required, but with a theoretical lag length/window width of four quarters, and standard unit root tests are also applied when testing for cointegration, given that the cointegration vector of (1,-1) is imposed in all cases.

[Table 1 here]

The first point of note is that the ANS Level coefficient $\beta_1(t)$ and its annual average $\beta_{1,t}$ are highly persistent, in that they do not reject the unit root hypothesis. That matches the persistence of the macroeconomic variables, given that steady-state output growth $\Delta Y^*(t)$, steady-state inflation $\Delta P^*(t)$, and their sum $[\Delta Y^*(t) + \Delta P^*(t)]$ also reject the hypothesis of a unit root. The first differences of $\beta_1(t)$, $\beta_{1,t}$, $\Delta Y^*(t)$, $\Delta P^*(t)$, and $[\Delta Y^*(t) + \Delta P^*(t)]$ all reject the unit root hypothesis, and so all have the same I(1) order of integration. This result confirms that the parsimonious random-walk specifications for the Level coefficient and the GCV steady-state variables within the ANS framework are appropriate/adequate representations of the data in practice.

The second point of note is that the hypothesis of $\beta_{1,t}$ and $[\Delta Y^*(t) + \Delta P^*(t)]$ being cointegrated with a (1,-1) vector is rejected. The analogous comparative tests for (1,-1) cointegration between $\beta_{1,t}$ and $\Delta P^*(t)$, AGS3_t and $[\Delta Y^*(t) + \Delta P^*(t)]$, and AGS3_t and $\Delta P^*(t)$ are also rejected. An inspection of the time series $\beta_{1,t} - [\Delta Y^*(t) + \Delta P^*(t)]$ in figure 6 shows that the deviations between $\beta_{1,t}$ and $[\Delta Y^*(t) + \Delta P^*(t)]$ have averaged around zero for the entire sample, but there was a highly persistent break from around the late 1970s/early 1980s. The latter is not surprising given the context of substantial change in the US economic and financial environment during the late-1970s to the mid-1980s. For example, a significant economic change was the Federal Reserve's Volcker-led disinflation from October 1979, and the subsequent maintenance of low inflation without the gold standard. Significant financial changes were progressive market deregulation, including eliminating interest rate restrictions and rationalising reserve requirements, and an increasing role for securitisation. Prior empirical work also mentions these reasons when establishing and documenting structural breaks between 1979:Q4 to 1984:Q1.[13]

The theoretical structure of the ANS framework would attribute the break to changes in the volatility and/or term premium for the Level coefficient, which in turn should correspond with changes to the risk parameters for steady-state growth plus steady-state inflation; i.e $\sigma_1^2 = \sum_{j=1}^{2J} \sigma_{0,j}^2$ and $\sigma_1 \rho_1 = \sum_{j=1}^{2J} \sigma_{0,j} \rho_{0,j}$ respectively (changes in the inflation variance σ_P^2 may also contribute). The parsimony of the ANS framework also offers a convenient means for establishing and documenting those changes; i.e formally test for structural breaks in the time series $\beta_{1,t} - [\Delta Y^*(t) + \Delta P^*(t)]$ based on supporting evidence from changes in the economic and financial environments that prevailed at the time, and re-estimate the ANS model over each of those periods to confirm that the volatilties and/or market prices of risk do change materially. However, such an investigation is beyond the scope of the simple illustration in this article, and remains to be undertaken in future work.

Regarding the non-Level coefficients, the ANS Slope and Bow coefficients are mean reverting, in that they each strongly reject the unit root hypothesis. That property carries through to $[\beta_2(t) \cdot q_2(1) + \beta_3(t) \cdot q_3(1)]$, and is consistent with the mean reversion in output growth $\Delta Y(t)$ and output growth less potential output growth, i.e $\Delta X(t) = \Delta Y(t) - \Delta Y^*(t)$, given those series also strongly reject the unit root hypothesis.

Table 2 contains the results from estimating the regression in equation 13 and the analogous comparative regressions. All of the estimated standard errors are adjusted using the Newey and West (1987) technique with a window width of three quarters to correct for the effect of MA(3) autocorrelation induced in $\varepsilon(t+1)$ due to the use of intertemporal data with an annual horizon at a quarterly frequency.

[Table 2 here]

The first point of note is that the regression in equation 13 shows a strong relationship between $\Delta X(t)$ and lagged $[\beta_2(t) \cdot q_2(1) + \beta_3(t) \cdot q_3(1)]$, with highly significant estimates of the coefficient $\alpha_{1,1}$. Similar results are obtained for the analogous comparative regressions of $\Delta Y(t)$ on lagged $[\beta_2(t) \cdot q_2(1) + \beta_3(t) \cdot q_3(1)]$, $\Delta X(t)$ on lagged [GS10 (t)-TB3 (t)], and $\Delta Y(t)$ on lagged [GS10 (t)-TB3 (t)]. The latter reproduces the standard result in the literature that [GS10 (t)-TB3 (t)] correlates strongly with future output growth, but as suggested by the ANS framework, the correlation with $\Delta X(t)$ is even stronger.

The second point of note is the rejection of the hypothesis that $\alpha_{1,1}$ has the predicted value of 1. An inspection of figure 8 shows evidence of a structural change for the relationship between $\Delta X(t)$ and lagged $[\beta_2(t) \cdot q_2(1) + \beta_3(t) \cdot q_3(1)]$ from around the late 1970s/early 1980s, and the context for that has already been discussed above. The ANS framework would suggest that the change is due to changes in the volatilties and/or the market prices of risk for the Slope and Bow coefficients, which should in turn correspond with changes to the risk parameters for output growth plus inflation relative to their steady-state values; i.e $\sum_{n=2}^{3} \sigma_n^2 \cdot h_2(\phi, m) \simeq -\sum_{j=1}^{2J} \frac{1}{2} \left[\sigma_{1,j}^2 - \sigma_{0,j}^2\right] \cdot [B_j(m)]^2$ and $\sum_{n=2}^{3} \gamma_n \cdot g_n(\phi, m) \simeq \sum_{j=1}^{2J} \left[\sigma_{1,j}\rho_{1,j} - \sigma_{0,j}\rho_{0,j}\right] \cdot B_j(m)$ respectively. Once again, the ANS

[13] For example, based on statistical tests for unknown breakpoints, Estrella et al. (2003) identifies structural breaks in October 1979 and October 1982 when the yield curve is used as an indicator of future inflation, and in September 1983 when the yield curve is used as an indicator of future output. Using a similar technique, Aïssa and Jouini (2003) documents a break in the inflation process in June 1982, and Lai (2004) suggests a break between mid-1980 and early-1981. Jardet (2004) documents a break in the yield curve/output relationship in March 1984, and the yield curve forecasting application of Krippner (2006) implies a break in yield curve term premia between September 1979 and October 1982.

framework offers a convenient means for further investigation of this aspect, but that is beyond the scope of this article and remains to be undertaken in future work.

Finally, note that the estimate of $\alpha_{0,1}$ is significantly negative. This indicates that average term premia over the entire sample were materially positive; i.e the shape of the yield curve overstated future values of $\Delta X(t)$ on average.

6 Conclusions and potential applications

The ANS framework (i.e the ANS model with its macroeconomic foundation) developed in this article establishes a formal theoretical justification for the NS approach to modelling the yield curve with simple components of time-to-maturity allowing for Gaussian dynamics. At the same time, the ANS framework also provides a theoretically-rigorous yet parsimonious and intuitive macro-finance tool. The simple and practical application in this article provides a ready interpretation of a series of empirical results from the macro-finance literature that relate the level and slope of the yield curve to output growth and inflation. However, the empirical application of the ANS framework also highlights that historical relationships between the yield curve, output growth, and inflation have been influenced by time-varying potential growth and especially by occasional and highly persistent changes to term premia. Investigating the latter within the ANS framework remains as a topic for future work.

The ANS framework, or straightforward extensions, may be applied more generally as a tool within macro-finance where the user is satisfied that a minimal model containing just the core variables of interest rates, output growth, and inflation with Gaussian dynamics is sufficient. An obvious example is extracting implied market expectations of output growth and inflation from the yield curve, particularly long-term inflation expectations with appropriate allowances for occasional changes in term premia. Or in reverse, the ANS framework could be used to assess the sensitivity of yield curve changes, and so the risk for fixed interest portfolios, to unanticipated changes in output growth and/or inflation (or suggest optimal positions for active views on data surprises). And given that the ANS framework is intertemporally-consistent and arbitrage-free, it also offers a means of valuing and hedging some of the macroeconomic derivatives that have been suggested by Shiller (1993 and 2003), and that have been provided to the market over recent years; e.g see Frankel and O'Neill (2002), Chicago Mercentile Exchange (2005), and Goldman Sachs (2005).

A Deriving the GCV forward rate curve

This appendix proceeds in three sub-sections: (1) outlining the general relationship between the expected path of the short rate and the forward rate curve within the HJM framework;[14] (2) calculating the expected path of the short rate for the GCV economy; and (3) calculating the forward rate curve for the GCV economy via the HJM framework.

A.1 The HJM framework

HJM specifies the relationship between the forward rate curve and the short rate under the physical measure as:

[14] This sub-section is abbreviated from Krippner (2006) appendix A, but makes the important clarification that the notation is based on time and time-to-maturity, as used for the market models originally introduced in Brace, Gatarek and Musiela (1997) (hereafter BGM), rather than the HJM time and time-of-maturity notation. Specifically, $f(t,m) = f_{\mathrm{BGM}}(t,m) = f_{\mathrm{HJM}}(t, t+m)$.

$$
\begin{aligned}
r(t+m) &= f(t,m)+\sum_{n=1}^{N}\int_0^m \sigma_n(v,m)\left\{\int_v^m \sigma_n(v,u)\,du\right\}dv \\
&\quad -\sum_{n=1}^{N}\int_0^m \sigma_n(v,m)\,\rho_n dv+\sum_{n=1}^{N}\int_t^{t+m}\sigma_n(t,m)\,dW_n(t) \qquad (14)
\end{aligned}
$$

where $r(t+m)$ is the short rate at time t as a function of future horizon m (so $t+m$ is a future point in time); $f(t,m)$ is the forward rate curve at time t as a function of time-to-maturity m; N is the number of independent stochastic processes that impart instantaneous random changes to the forward rate curve; $\sigma_n(v,m)$ is the volatility function for the process n; ρ_n is the market price of risk for the process n; $dW_n(v)$ are independent Wiener variables under the physical measure; and u and v are dummy integration variables.

Applying the expectations operator as at time t to equation 14 and rearranging gives:

$$
f(t,m)=E_t[r(t+m)]+\sum_{n=1}^{N}\int_0^m \sigma_n(v,m)\,\rho_n dv-\sum_{n=1}^{N}\int_0^m \sigma_n(v,m)\left\{\int_v^m \sigma_n(v,u)\,du\right\}dv \qquad (15)
$$

where $E_t[r(t+m)]$ is the expected path of the short rate as at time t as a function of future horizon m.

A.2 The GCV expected path of the short rate

Heuristically, the expected path of the short rate for the GCV economy may be obtained by applying the expectations operator to equation 4 and solving the resulting ordinary differential equation in future time m. That is, $E_t[ds_j(t+m)]=-\kappa_j\{E_t[ds_j(t+m)]-\theta_j(t)\}\,dm$, which has the solution $E_t[s_j(t+m)]=\theta_j(t)+[s_j(t)-\theta_j(t)]\cdot\exp(-\kappa_j m)$, and summing over all j reproduces equation 6.

However, the presence of the stochastic process for the steady-state variables makes the calculation more subtle.[15] Hence, with reference to equation 4, define a point u so that $t<u<t+m$ and $ds_j(u)=-\kappa_j[s_j(u)-\theta_j(u)]\,du+\sigma_{1,j}dz_{1,j}(u)$. This may be re-arranged as $ds_j(u)+\kappa_j s_j(u)\,du=\kappa_j\theta_j(u)\,du+\sigma_{1,j}dz_{1,j}(u)$ and expressed as $d[s_j(u)\cdot\exp(\kappa_j u)]=\kappa_j\theta_j(u)\cdot\exp(\kappa_j u)\,du+\sigma_{1,j}\cdot\exp(\kappa_j u)\,dz_{1,j}(u)$. Integrating from t to $t+m$ and taking the result for the lower limit of integration to the right-hand side gives the result:

$$
\begin{aligned}
s_j(t+m)\cdot\exp(\kappa_j[t+m]) &= s_j(t)\cdot\exp(\kappa_j t)+\kappa_j\int_t^{t+m}\theta_j(u)\cdot\exp(\kappa_j u)\,du \\
&\quad +\int_t^{t+m}\sigma_{1,j}\cdot\exp(\kappa_j u)\,dz_{1,j}(u) \qquad (16)
\end{aligned}
$$

To evaluate $\kappa_j\int_t^{t+m}\theta_j(u)\cdot\exp(\kappa_j u)\,du$, first note that equation 4 defines $d\theta_j(v)=\sigma_{0,j}dz_{0,j}(v)$. Integrating from t to u and taking the result for the lower limit of integration to the right-hand side gives $\theta_j(u)=\theta_j(t)+\sigma_{0,j}\int_t^u dz_{0,j}(v)$. Substituting this result into $\kappa_j\int_t^{t+m}\theta_j(u)\cdot\exp(\kappa_j u)\,du$ and expanding gives $\kappa_j\theta_j(t)\int_t^{t+m}\exp(\kappa_j u)\,du+\kappa_j\sigma_{0,j}\int_t^{t+m}\exp(\kappa_j u)\left[\int_t^u dz_{0,j}(v)\right]du$. The first integral is $\theta_j(t)[\exp(\kappa_j u)]_t^{t+m}=\theta_j(t)[\exp(\kappa_j[t+m])-\exp(\kappa_j t)]$, and the second integral is $\kappa_j\sigma_{0,j}\int_t^{t+m}\left(\int_v^{t+m}\exp(\kappa_j u)\,du\right)dz_{0,j}(v)$, where the stochastic Fubini theorem has been used to reverse the sequence of integration. The inner integral is $\frac{1}{\kappa_j}[\exp(\kappa_j[t+m])-\exp(\kappa_j v)]$, and so the final expression of $\kappa_j\int_t^{t+m}\theta_j(u)\cdot\exp(\kappa_j u)\,du$ is $\theta_j(t)[\exp(\kappa_j[t+m])-\exp(\kappa_j t)]+\sigma_{0,j}\int_t^{t+m}[\exp(\kappa_j[t+m])-\exp(\kappa_j v)]\,dz_{0,j}(v)$.

[15] I thank Carl Chiarella for pointing this out, and for detailing how the calculation should be approached.

Substituting that result into equation 16 and factoring out $\exp\left(\kappa_j\left[t+m\right]\right)$ across the entire equation gives the final expression for $s_j\left(t+m\right)$ and its dynamics, i.e:

$$\begin{aligned} s_j\left(t+m\right) &= s_j\left(t\right)\cdot\exp\left(-\kappa_j m\right)+\theta_j\left(t\right)\cdot\left[1-\exp\left(-\kappa_j m\right)\right] \\ &\quad +\sigma_{0,j}\int_t^{t+m} dz_{0,j}\left(v\right)-\sigma_{0,j}\int_t^{t+m}\exp\left(-\kappa_j\left[t+m-v\right]\right)dz_{0,j}\left(v\right) \\ &\quad +\sigma_{1,j}\int_t^{t+m}\exp\left(-\kappa_j\left[t+m-u\right]\right)dz_{1,j}\left(u\right) \end{aligned} \tag{17}$$

Applying the expectations operator as at time t then gives $E_t\left[s_j\left(t+m\right)\right]=\theta_j\left(t\right)+\left[s_j\left(t\right)-\theta_j\left(t\right)\right]\cdot\exp\left(-\kappa_j m\right)$, and summing over all j reproduces equation 6.

Regarding equation 7, $E_t\left[r\left(t+m\right)\right]=E_t\left[\sum_{j=1}^{2J}s_j\left(t+m\right)\right]=E_t[dY\left(t+m\right)+dP\left(t+m\right)-\sigma_P^2]=E_t\left[dY^*\left(t+m\right)+dP^*\left(t+m\right)-\sigma_P^2\right]+E_t\left[dX\left(t+m\right)\right]$, where $E_t\left[dX\left(t+m\right)\right]$ is defined in Proposition 1. Then note that $E_t[dY^*\left(t+m\right)+dP^*\left(t+m\right)-\sigma_P^2]=E_t\left[\sum_{j=1}^{2J}\theta_j\left(t+m\right)\right]=E_t\left[\sum_{j=1}^{2J}\theta_j\left(t\right)+\sigma_{0,j}\int_t^{t+m}dz_{0,j}\left(v\right)\right]=\sum_{j=1}^{2J}\theta_j\left(t\right)=dY^*\left(t\right)+dP^*\left(t\right)-\sigma_P^2$. Hence, $E_t\left[r\left(t+m\right)\right]=dY^*\left(t\right)+dP^*\left(t\right)-\sigma_P^2+E_t[dX\left(t+m\right)]$.

A.3 The GCV forward rate curve

The stochastic terms in equation 17 define how innovations in each GCV factor will impart instantaneous random changes to the expected path of the short rate. Equation 15 shows those innovations will be simultaneously reflected in the forward rate curve, and that defines the volatility functions required for the HJM framework calculations.

Firstly, $\sigma_{0,j}\int_t^{t+m}dz_{0,j}\left(v\right)$ from equation 17 shows that an innovation $dz_{0,j}\left(t\right)$ will result in an instantaneous parallel shift of $\sigma_{0,j}\cdot dz_{0,j}\left(t\right)$ to $f\left(t,m\right)$. Therefore, this contribution to the volatility function for factor j is $\sigma_n\left(t,m\right)=\sigma_{0,j}$, which gives the first equation 15 integral $\int_0^m\sigma_{0,j}\rho_{0,j}dv=\sigma_{0,j}\rho_{0,j}\cdot\left[v\right]_0^m=\sigma_{0,j}\rho_{0,j}\cdot m$, and the second equation 15 integral $\int_0^m\sigma_{0,j}\cdot\left\{\int_v^m\sigma_{0,j}du\right\}dv=\int_0^m\sigma_{0,j}\cdot\left\{\sigma_{0,j}\cdot\left[u\right]_v^m\right\}dv=\int_0^m\sigma_{0,j}^2\cdot\left[m-v\right]dv=\sigma_{0,j}^2\cdot\left[mv-\frac{v^2}{2}\right]_0^m=\frac{1}{2}\sigma_{0,j}^2\cdot m^2$.

Secondly, the third line of equation 17 shows that an innovation $dz_{1,j}\left(t\right)$ will result in an instantaneous non-parallel shift of $\sigma_{1,j}\cdot\exp\left(-\kappa_j m\right)\cdot dz_{1,j}\left(t\right)$ (i.e an exponential decay function by time-to-maturity m) to $f\left(t,m\right)$. Therefore, this contribution to the volatility function for factor j is $\sigma_n\left(t,m\right)=\sigma_{1,j}\cdot\exp\left(-\kappa_j m\right)$, which gives the first equation 15 integral $\int_0^m\sigma_{1,j}\cdot\exp\left(-\kappa_j v\right)\cdot\rho_{1,j}dv=\sigma_{1,j}\rho_{1,j}\cdot\left[-\frac{1}{\kappa_j}\exp\left(-\kappa_j v\right)\right]_0^m=\sigma_{1,j}\rho_{1,j}\cdot B_j\left(m\right)$ where $B_j\left(m\right)=\frac{1}{\kappa_j}\left[1-\exp\left(-\kappa_j m\right)\right]$. The second equation 15 integral is calculated in two steps, i.e: $\int_v^m\sigma_n\left(v,u\right)du=\int_s^m\sigma_{1,j}\cdot\exp\left(-\kappa_j\left[u-v\right]\right)du=\sigma_{1,j}\cdot\left[-\frac{1}{\kappa_j}\exp\left(-\kappa_j\left[u-v\right]\right)\right]_v^m=\frac{\sigma_{1,j}}{\kappa_j}\cdot\left[1-\exp\left(-\kappa_j\left[m-v\right]\right)\right]$. Then $\int_0^m\sigma_n\left(v,m\right)\left\{\int_v^m\sigma_n\left(v,u\right)du\right\}dv$ is calculated as: $\int_0^m\sigma_{1,j}\cdot\exp\left(-\kappa_j\left[m-v\right]\right)\cdot\frac{\sigma_{1,j}}{\kappa_j}\cdot\left[1-\exp\left(-\kappa_j\left[m-v\right]\right)\right]dv=\frac{\sigma_{1,j}^2}{\kappa_j}\cdot\int_0^m[\exp\left(-\kappa_j\left[m-v\right]\right)-\exp\left(-2\kappa_j\left[m-v\right]\right)]dv=\frac{\sigma_{1,j}^2}{\kappa_j^2}\cdot\left[\exp\left(-\kappa_j\left[m-v\right]\right)-\frac{1}{2}\exp\left(-2\kappa_j\left[m-v\right]\right)\right]_0^m=\frac{\sigma_{1,j}^2}{\kappa_j^2}\cdot\left[1-\exp\left(-\kappa_j m\right)-\frac{1}{2}+\frac{1}{2}\exp\left(-2\kappa_j m\right)\right]=\frac{\sigma_{1,j}^2}{2\kappa_j^2}\cdot\left[1-\exp\left(-\kappa_j m\right)+\exp\left(-2\kappa_j m\right)\right]=\frac{\sigma_{1,j}^2}{2\kappa_j^2}\cdot\left[1-\exp\left(-\kappa_j m\right)\right]^2=-\frac{1}{2}\sigma_{1,j}^2\cdot\left[B_j\left(m\right)\right]^2$

Thirdly, the second component on line 2 of equation 17 shows that innovations in $dz_{0,j}\left(t\right)$ will also result in an instantaneous non-parallel shift of $-\sigma_{0,j}\cdot\exp\left(-\kappa_j m\right)\cdot dz_{0,j}\left(t\right)$ to $f\left(t,m\right)$, in addition to the parallel shift already noted earlier. The integrals for these non-parallel components follow those for $dz_{1,j}\left(t\right)$ above, giving the results $-\sigma_{0,j}\rho_{0,j}\cdot B_j\left(m\right)$ and $\frac{1}{2}\sigma_{1,j}^2\cdot\left[B_j\left(m\right)\right]^2$.

Substituting $E_t\left[r\left(t+m\right)\right]$ from section A.2 and the calculations from this section into equation 15 gives equation 8. Equation 9 follows from a straightforward substitution of the results from the last paragraph in appendix A.2.

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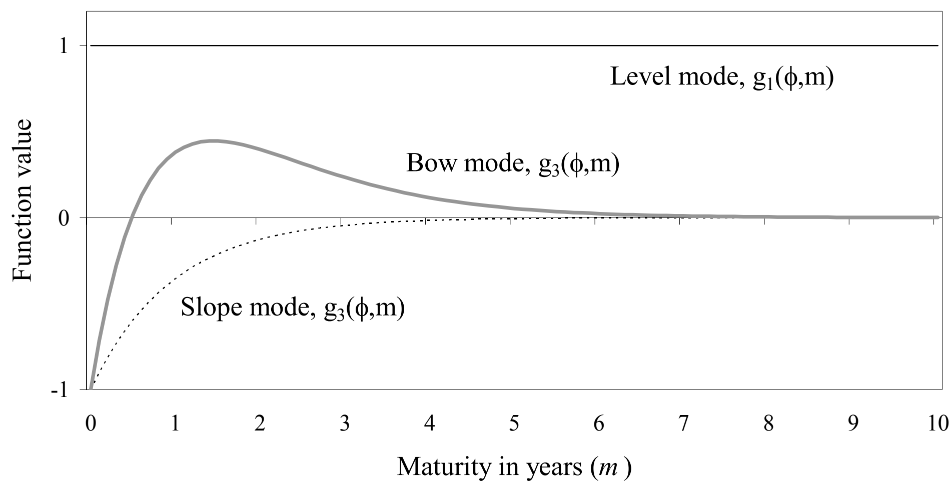

Figure 1: The three modes used to represent the forward rate curve $f(t,m)$ in NS models. This illustration uses $\phi = 1.03$.

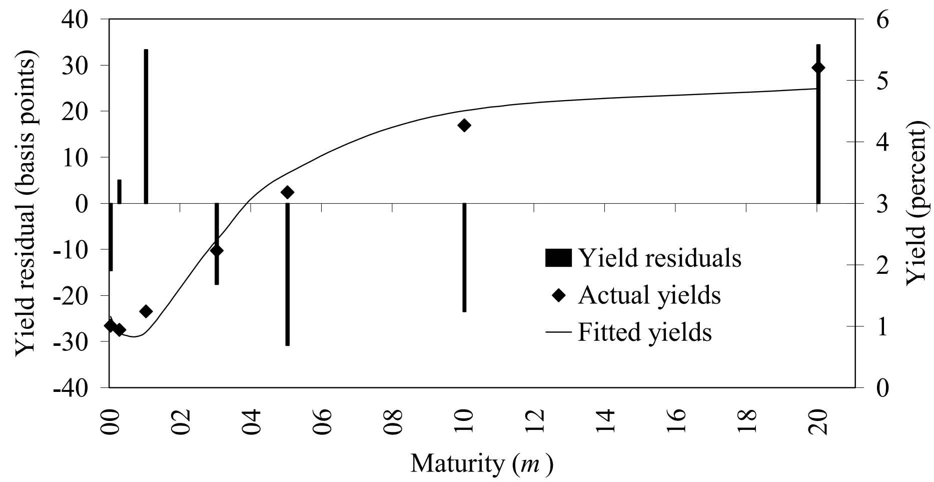

Figure 2: Yield curve data for the month of September 2003 and the fitted yields based on the estimated ANS model. The estimated Level, Slope, and Bow coefficients are, respectively, $\beta_1\,(\text{Sep-03}) = 5.96\%$, $\beta_2\,(\text{Sep-03}) = 8.67\%$, and $\beta_3\,(\text{Sep-03}) = -3.90\%$. The ANS parameters are $\phi = 1.03$, $\rho_1 = 2.66\%$, $\sigma_1 = 0.77\%$, $\sigma_2 = 2.26\%$, and $\sigma_3 = 1.74\%$.

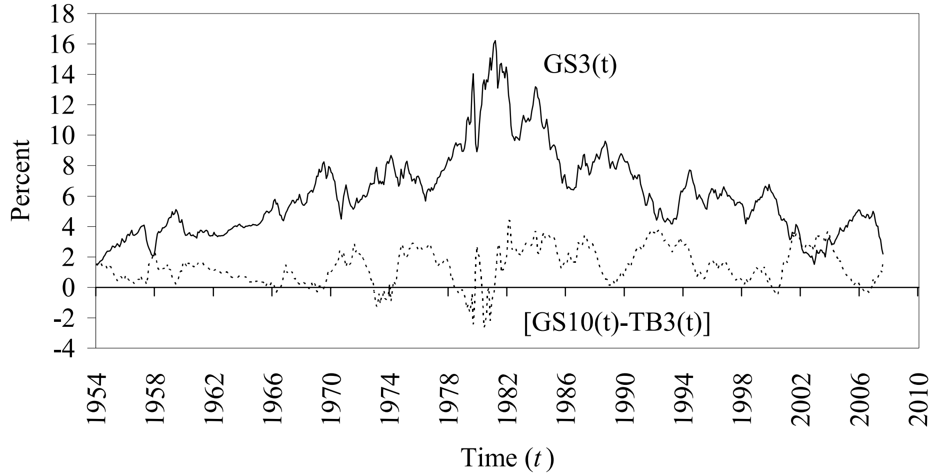

Figure 3: The 3-year government bond yield, GS3(t), and the slope of the yield curve, denoted [GS10(t)-TB3(t)], as measured by the spread between the 10-year government bond yield, GS10(t), and the 3-month Treasury bill rate, TB3(t).

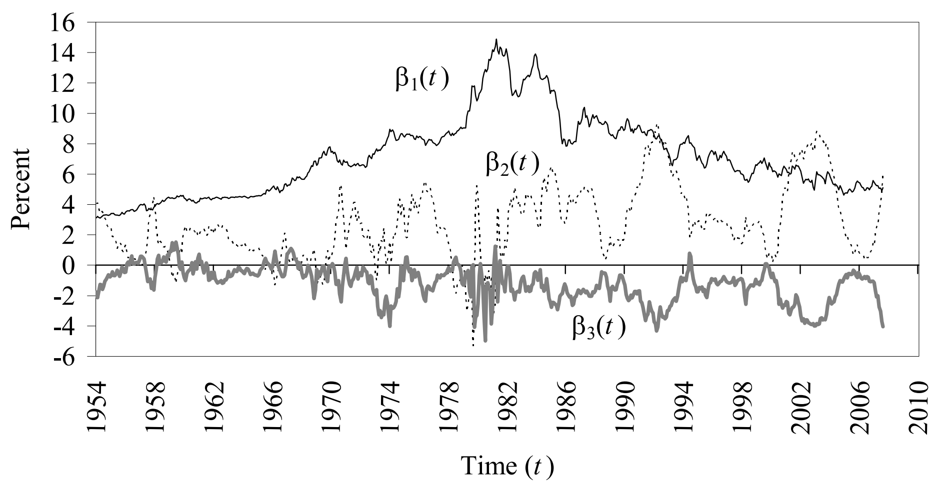

Figure 4: The time series of the estimated ANS coefficients, i.e Level $\beta_1(t)$, Slope $\beta_2(t)$, and Bow $\beta_3(t)$. The ANS coefficients at each point in time are estimated using the seven points of yield curve data observed at that point in time, as in the example for September 2003 illustrated in figure 2.

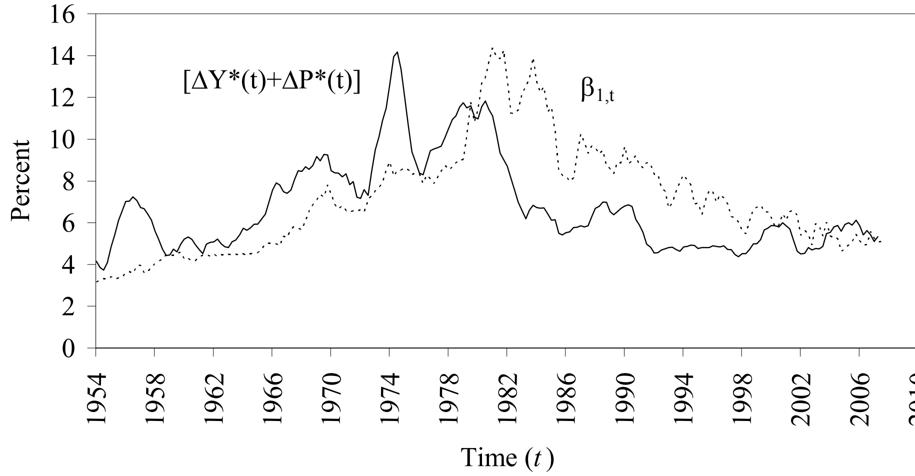

Figure 5: The time series of the annual average Level coefficient $\beta_{1,t}$ and annual potential output growth plus annual inflation $[\Delta Y^*(t) + \Delta P^*(t)]$.

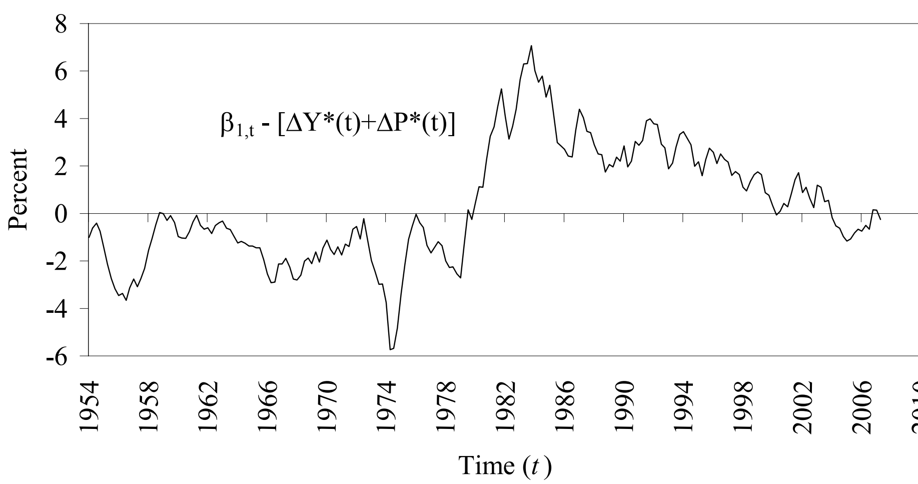

Figure 6: The time series of the annual average Level coefficient $\beta_{1,t}$ less annual potential output growth plus annual inflation $[\Delta Y^*(t) + \Delta P^*(t)]$.

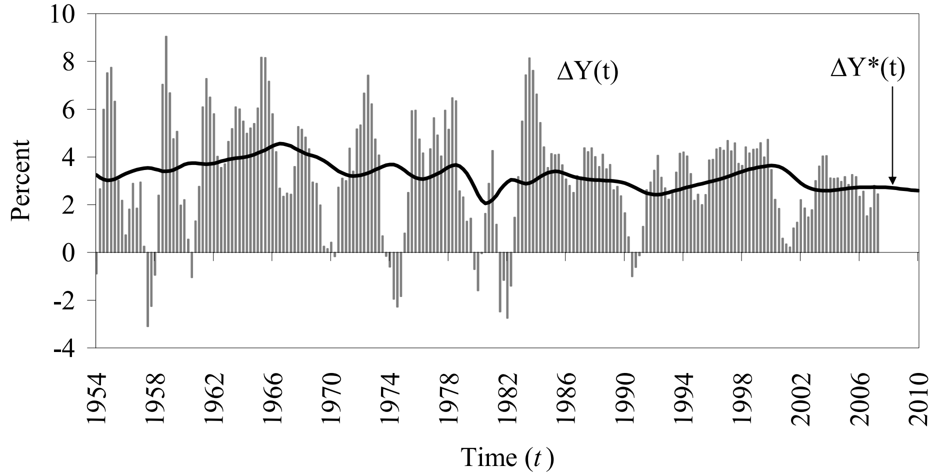

Figure 7: The time series of annual GDP growth, $\Delta Y\,(t)$, and the annual change in Congressional Budget Office potential GDP, $\Delta Y^*\,(t)$. These data are used to construct the $\Delta X(t) = \Delta Y\,(t) - \Delta Y^*\,(t)$ data plotted in figure 8.

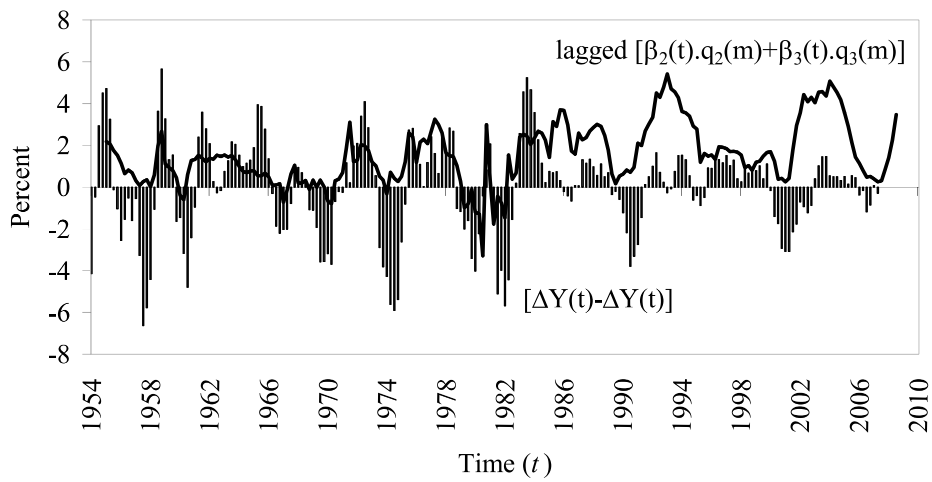

Figure 8: The time series of $\Delta X\,(t) = \Delta Y\,(t) - \Delta Y^*\,(t)$ based on the $\Delta Y\,(t)$ and $\Delta Y^*\,(t)$ data plotted in figure 7, and $[\beta_2\,(t) \cdot q_2\,(1) + \beta_3\,(t) \cdot q_3\,(1)]$ lagged four quarters.

Time series \ unit root tests	Test on level with 3Q lag/window ADF	PP	Test on level with selected lag/window ADF	PP	First difference test with 4Q lag/window ADF	PP
Level coefficient, $\beta_1(t)$	-1.93	-1.75	-1.64	-1.67	-7.07 ***	-13.15 ***
Slope coefficient, $\beta_2(t)$	-3.48 ***	-3.97 ***	-4.13 ***	-4.33 ***	n/a	n/a
Bow coefficient, $\beta_3(t)$	-3.40 **	-5.72 ***	-3.64 ***	-6.47 ***	n/a	n/a
Annual average Level coefficient, $\beta_{1,t}$	-1.55	-1.57	-1.74	-1.59	-4.29 ***	-5.31 ***
Steady-state output growth, $\Delta Y^*(t)$	-1.29	-1.79	-1.03	-1.93	-3.66 ***	-4.74 ***
Steady-state inflation, $\Delta P^*(t)$	-2.66 *	-2.07	-1.51	-2.10	-6.74 ***	-9.12 ***
Steady-state output growth plus inflation, $[\Delta Y^*(t)+\Delta P^*(t)]$	-2.72 *	-2.09	-1.41	-2.06	-6.73 ***	-8.85 ***
$\beta_{1,t}$ - $[\Delta Y^*(t)+\Delta P^*(t)]$	-2.43	-1.83	-1.81	-1.87	-5.56 ***	-7.00 ***
$\beta_{1,t}$ - $\Delta P^*(t)$	-2.52	-1.93	-2.10	-1.96	-5.68 ***	-7.22 ***
Annual average 3-year bond yield, $AGS3_t$	-2.02	-1.91	-1.82	-1.95	-4.70 ***	-5.44 ***
$AGS3_t$ - $[\Delta P^*(t)+\Delta P^*(t)]$	-2.46	-2.06	-2.32	-1.99	-6.13 ***	-7.09 ***
$AGS3_t$ - $\Delta P^*(t)$	-2.60 *	-2.20	-2.74 *	-2.12	-6.23 ***	-7.14 ***
Output growth, $\Delta Y(t)$	-7.24 ***	-5.68 ***	-3.29 **	-4.68 ***	n/a	n/a
$\Delta X(t) = \Delta Y(t) - \Delta Y^*(t)$	-8.00 ***	-5.92 ***	-4.30 ***	-5.02 ***	n/a	n/a
Lagged $\beta_2(t).q_2(1)+\beta_3(t).q_3(1)$	-3.59 ***	-3.99 ***	-4.11 ***	-4.32 ***	n/a	n/a
Lagged 10-year yield less 3-month rate [GS10(t)-TB3(t)]	-4.53 ***	-4.88 ***	-3.29 **	-5.07 ***	n/a	n/a

Table 1: Unit root tests with an estimated constant. ADF is augmented Dickey-Fuller, and PP is Phillips-Perron. ***, **, * respectively represent 1, 5, and 10 percent levels of significance based on the critical values -3.46, -2.88, and -2.57 from Hamilton (1994) p.763, Case 2.

Regression \ results	R^2	$\alpha_{0,1}$	s.e $\alpha_{0,1}$	$\alpha_{1,1}$	s.e $\alpha_{1,1}$	$\alpha_{1,1}$ - 1
$\Delta X(t) = \Delta Y(t) - \Delta Y^*(t)$ on lagged $[\beta_2(t).q_2(1)+\beta_3(t).q_3(1)]$	0.189	-1.07 ***	0.35	0.67 ***	0.15	-0.33 **
$\Delta X(t)$ on lagged [GS10(t)-TB3(t)]	0.229	-1.24 ***	0.35	0.89 ***	0.17	n/a
$\Delta Y(t)$ on lagged $[\beta_2(t).q_2(1)+\beta_3(t).q_3(1)]$	0.108	2.42 ***	0.39	0.52 ***	0.16	n/a
$\Delta Y(t)$ on lagged [GS10(t)-TB3(t)]	0.153	2.22 ***	0.38	0.74 ***	0.18	n/a

Table 2: Results of estimating equation 13 and the analogous comparative regressions. ***, **, * respectively represent 1, 5, and 10 percent two-tailed levels of significance based on the t-statistics α/s.e α with 208 degrees of freedom.