Collusion via Resale*

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Abstract

The English auction is susceptible to tacit collusion when post-auction inter-bidder resale is allowed. We show this by constructing a continuum of equilibria where, with positive probability, one bidder wins the auction without any competition and divides the spoils by optimally reselling the good to the other bidders. Such equilibria support a collusive bidding pattern without requiring the colluders to make *any* commitment on bidding behavior or post-bidding spoil-division. The equilibria are valid for any number of asymmetric or symmetric bidders, arbitrary reserve prices, and various resale market rules. In symmetric environments, these equilibria interim Pareto dominate (among bidders) the standard value-bidding equilibrium.

1 Introduction

In private-value English auctions that ban resale, it is a dominant strategy for each participant to bid up to her use value. With resale allowed, value-bidding remains an equilibrium outcome, but there is no dominant strategy. Resale opens the possibility that some bidders will optimally drop out at a price below their use values. They prefer to let a competitor win and buy from her in the resale market. The existence of non-value-bidding equilibria is

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important because the celebrated advantages of the English auction, in particular efficiency, are based on value-bidding, and because resale is possible in most applications.

In this paper we construct a family of non-value-bidding equilibria for an English auction that allows inter-bidder resale, and we prove that such equilibria exist in any independent private value environment (symmetric or asymmetric) for any number of bidders (Proposition 1). Each equilibrium in this family is identified by the choice of a *speculating* bidder and a cutoff type, below which all bidders, except the speculating bidder, bid zero. All bidders with types above the cutoff bid up to their values. In cases where the speculating bidder wins the initial auction and has a sufficiently low type, she will offer the item for resale instead of consuming it. Because the determination of the speculator does not depend on her type and the resale market retains information asymmetry the final outcome may be inefficient.

Equilibria with a speculating bidder extract additional surplus from the seller, and hence provide an opportunity for a form of tacit collusion among the bidders. Provided that bidders are not too asymmetric exante, by using a publicly observed randomizing device (or sunspot) to select the speculator, the bidders can distribute the surplus in a way that makes every bidder of every type better off than under the value-bidding equilibrium.[1] Furthermore, the spoil is divided through a continuation equilibrium in the resale market instead of relying on any pre-auction agreement among the colluders that may require post-auction enforcements. Thus, resale facilitates noncooperative collusive equilibria that interim Pareto dominate (for bidders) the standard value-bidding equilibrium (Proposition 2).

Blume and Heidhues (2004) construct equilibria in dominated strategies for English auctions without resale that are similar to our equilibria: a designated bidder bids the cutoff value if her type is below it, and all other bidders with types below the cutoff bid 0. The similarity suggests that if a public randomization device is used to determine the designated high bidder, then one can obtain a collusive equilibrium in the no-resale model that interim Pareto dominates the value-bidding equilibrium, as in our resale model. However, this is not generally true. For a large class of value distributions, including all strictly concave distribu-

[1]Readers who are familiar with U.S. litigation history might draw some parallels between our proposed use of a sunspots variable and the famous phases-of-the-moon bidding ring that was operated by electrical equipment suppliers in the 1950s. However, despite some reports, the phases-of-the-moon scheme earned its designation because it involved an explicit two-week rotation to determine the low bidder. While it perhaps could have been, bidding was not actually determined by the phase of the moon. See Smith (1961).

tions, high-value bidders are worse off under collusion if resale is prohibited (Proposition 3). Resale opportunities allow bidders to realize additional surplus that is distributed among bidders through the resale equilibria in a way that makes collusion preferable to all bidder types. In this sense, resale is essential for collusion.

No binding agreement among the colluding bidders is required in the collusive equilibria we describe. The recommendation made by the sunspot device is not binding. Rather, the sunspot plays the role of a correlating device in a correlated equilibrium (however private recommendations are not required). Once the sunspot identifies the speculating bidder it is in the interest of each bidder to bid accordingly in the initial auction based on the belief that others will follow their assigned roles. After the initial auction, the colluding bidders optimally carry out their final allocation through the resale market, without relying on any form of enforcement.

McAfee and McMillan (1992) identify four obstacles to successful collusion. They point out that the most significant factor in the downfall of many cartels is the issue of "dividing the spoils." Here this is resolved fairly, in a way that is acceptable to all, by the sunspot device.[2] The second obstacle, enforcement, is also avoided here since the participants obtain collusive payoffs through fully rational, noncooperative equilibrium behavior.[3] The fact that formal agreements and enforcement are not required mitigates the destructive impact of new entrants, which is the third obstacle described by McAfee and McMillan. Finally, the equilibria are fairly robust to the introduction of positive reserve prices.[4] Hence, actions taken by the seller to destabilize collusion, the final obstacle raised by McAfee and McMillan, need not eliminate collusive equilibria.

[2]Dividing the spoils is in fact the main issue addressed by McAfee and McMillan (1992), whose solution also involves randomization. They describe an equilibrium for a first-price auction in which bidders submit identical bids and allow the auctioneer to randomly determine the winner.

[3]The simplest collusive mechanisms (eg. Robinson, 1985) require some type of enforcement or punishment to prevent bidders from cheating on the collusive agreement. More elaborate schemes, cf. Graham and Marshall (1987), McAfee and McMillan (1992), and Mailath and Zemsky (1991), as well as the more recent literature on optimal auctions given collusion such as Laffont and Martimort (2000) and Che and Kim (2006), are incentive compatible. However, they require that bidders participate in some type of pre-auction side mechanism, selected by an external mediator, to form binding agreements.

[4]The equilibrium construction generalizes to arbitrary reserve prices. The collusion result holds for sufficiently small reserve prices. For environments with uniformly distributed use values, we have verified that the collusion result holds for arbitrary reserve prices in markets with two bidders and in markets with sufficiently many bidders.

The cutoff-bidding strategy in our equilibria is built upon Garratt and Tröger (2006) for English and second-price auctions. However, there are nontrivial differences. In Garratt and Tröger, the player who will become the reseller has no private information, and the other players have identical prior distributions. In contrast, in this paper every player has private information and players are asymmetric, i.e., they may have different prior distributions, and moreover we consider a wider range of resale market structures. Our extension of the equilibrium construction to the case of asymmetric bidders is made possible by conditioning the speculating bidder's bids on the identities of people who stay in the auction. This information is not available in a sealed-bid format. Hence, our equilibrium construction only works for a second-price auction when the players are symmetric (see Remark 3 following Proposition 1).[5]

In addition to allowing any number of asymmetric bidders,[6] the main value-added of this paper to the auction-resale literature is that it allows a variety of rules that govern how the bargaining power is distributed among the n bidders during resale.[7] That includes giving all the bargaining power to the reseller (Zheng (2002)), giving all the bargaining power to the buyers at resale (Haile (2003)), convex combinations between the two (Calzolari and Pavan (2006)), and English or second-price auctions given a reserve price chosen by the reseller. We obtain such generality of the resale markets by proving novel, general comparative statics properties for the continuation equilibria at resale (Section 3).

Recent works by Lebrun (2007) and Hafalir and Krishna (2007) compare revenue in first- and second-price auctions with resale. Hafalir and Krishna show that in 2-bidder, asymmetric auctions there exists a "general revenue ranking" in favor of first-price auctions, provided bidders play the value-bidding equilibrium in the second price auction. Lebrun shows that this ranking does not necessarily hold when behavior (mixed) strategies are allowed.[8] The existence of the additional equilibria described in this paper does not change

[5]For second-price auctions with symmetric bidders, our construction generalizes the counterpart in Garratt and Tröger (2006) to the case where the speculator has a private value.

[6]Except Zheng (2002), the auction-resale literature assumes either that bidders are symmetric or that there are only two bidders.

[7]To keep the model closed, we maintain the assumption that a resale mechanism is either exogenous or optimally chosen by one of the n bidders. Hence the colluding bidders cannot rely on a neutral mediator to select for them an incentive efficient mechanism to dissolve their partnership efficiently (e.g., Laffont and Martimort (2000), Cramton, Gibbons, and Klemperer (1987)).

[8]For the case where two bidders have different value distributions Lebrun (2007) shows that either auction

the revenue ranking established by Hafalir and Krishna, since the revenue generated by any of our equilibria is no greater than the revenue received by the seller in the value-bidding equilibrium.

Collusive equilibria have been constructed for multi-unit auctions by Milgrom (2000), Brusco and Lopomo (2002) and Engelbrecht-Wiggans and Kahn (2005), however resale does not play a role.[9] In these multi-unit environments bidders signal their preferences in early rounds and then optimally abstain from bidding on other bidders' preferred items. Interestingly, the open aspect of the ascending English auction is essential in their construction, as it is here in the case of ex-ante asymmetric bidders.

2 Model

We consider environments with $n \geq 2$ risk-neutral bidders pursuing a single indivisible private good. Bidder $i \in N := \{1, \ldots, n\}$ has a privately known use value, or *type*, $t_i \in T_i := [0, \bar{t}_i]$ ($\bar{t}_i > 0$) for the good. From the viewpoint of the other bidders, t_i is independently distributed according to a probability distribution with cumulative distribution function F_i. We assume that F_i has a density f_i that is positive and continuous on T_i and identically 0 elsewhere. We add the standard assumption that the hazard rates are weakly increasing; i.e., for all $i \in N$, the mapping

$$t \mapsto f_i(t)/(1 - F_i(t)) \ \text{ is weakly increasing on } \ [0, \bar{t}_i).$$

The *type space* is denoted by $\mathbf{T} := T_1 \times \cdots \times T_n$.[10] We consider a 2-period game, which begins after each bidder $i \in N$ has privately observed her use value $t_i \in T_i$.

In period 1, the good is offered via an English auction with zero reserve price. (Remark 1 after Proposition 1 will extend the result to positive reserve prices.) The auction winner either consumes the good in period 1, thereby ending the game, or becomes the *period-2 seller*, who offers the good in period 2 for resale to the losing bidders, called *period-2 buyers*.

format can have higher expected revenue; it depends on equilibrium selection in second-price auction.

[9] Pagnozzi (2007) analyzes multi-unit auctions with resale in a complete information model.

[10] We shall use boldface letters to denote multidimensional quantities.

The game ends when period 2 ends.

Our model allows a variety of alternative rules for resale in period 2:

I Myerson's auction: The period-2 seller commits to a selling mechanism, and resale is not allowed after the operation of the mechanism. (Remark 2 after Proposition 1 will partially extend our result to allow repeated resale as in Zheng (2002).)

II English or second-price auction with a reserve price uniform for all period-2 bidders: the only policy instrument for the period-2 seller is the reserve price.

III Haile's (2003) auction in the environments with $n > 2$: An English auction without reserve price and with the option for the period-2 seller to reject all the bids after bidders have all stopped raising their bids. We assume that bids in this auction start at zero.

IV Player-specific bargaining power in the 2-bidder environment, similar to Calzolari and Pavan (2006): for some exogenous $\theta_1, \theta_2 \in (0,1)$, with $\theta_1 + \theta_2 = 1$, bidder i makes a take-it-or-leave offer to the other bidder $-i$ with probability θ_i.

We also allow any combination of the above rules such that each rule is applied with a predetermined probability.

Every player's discount factor is $\delta \in (0,1]$. Given type t_i, the bidder i's expected payoff, from the viewpoint of period 1, is equal to

$$\delta^{s-1} t_i \mathbf{1}_{i=\text{owner}} - m_1 - \delta m_2,$$

where $\mathbf{1}_{i=\text{owner}}$ is the indicator function for the event that i is the final owner of the good at the end of the game, s is the period in which i consumes the good, and m_r $(r = 1, 2)$ is the expected value of the bidder's net monetary payment in period r.

2.1 Histories, strategies, and beliefs

During a period-1 auction a history is represented by $h := (h_i)_{i \in N} \in [0, \infty)^n$, where h_i is the highest price up to which bidder i has stayed in the auction. Let

$$\mathcal{H}_i := \left\{ h \in [0, \infty)^n \mid h_i = \max_{j \in N} h_j \text{ and } \exists\, k \neq i \text{ s.t. } h_k = h_i \right\}$$

denote the set of *nonterminal histories* in which bidder i and at least one other bidder remain active. The model we use for the period-1 English auction is the standard button auction (Milgrom and Weber, 1982) in which a bidder's only decision is when to withdraw irreversibly from the bidding. Accordingly, a *bidding strategy profile* $(\beta_i(\cdot \mid h))_{i \in N, h \in \mathcal{H}_i}$ determines, for any bidder $i \in N$, any nonterminal history $h \in \mathcal{H}_i$ and any type $t_i \in T_i$, the price $\beta_i(t_i \mid h)$ at which bidder i plans to drop out.

A *terminal history* records the actual dropout price of each losing bidder in the period-1 auction up to the end of the auction, as well as the identity of the period-1 winner. Let $\mathcal{H}$ denote the set of all possible terminal histories. A *resale decision profile* is a vector $(\gamma_h)_{h \in \mathcal{H}}$, where $\gamma_h(t) = 1$ if type t of the winning bidder offers the good for resale at the ending history h, and $\gamma_h(t) = 0$ if the winner consumes the good in period 1.

We restrict attention to equilibria where posterior beliefs remain stochastically independent across all players. Hence we model a *belief profile*, as a mapping that associates to each history of the game a product of stochastically independent probability distributions on $T_1, \ldots, T_n$.

The part of the belief profile that will play an important role in our analysis is the *post-auction belief profile*, denoted by $(\mathbf{G}_h)_{h \in \mathcal{H}}$. For any terminal history h, $\mathbf{G}_h$, called the *post-auction belief*, is the profile of independent distributions of the bidders' types that are held by bidders at the beginning of period 2 when the history of the period-1 auction is h and that the period-1 winner has decided to offer the good for resale. Since the history of an English auction is common knowledge, we may assume, without loss of generality, that the posterior beliefs, including $\mathbf{G}_h$, are common knowledge up to the start of period 2.

As there is no further resale after period 2 and the post-auction belief is commonly known, the revelation principle for period 2 holds. Thus, we directly formulate a *period-2 outcome* *function*

$$((P_{ij}, Q_{ij})_{i\in N\setminus\{j\}})_{j\in N}$$

such that, given any post-auction belief $\mathbf{J}$ (e.g., $\mathbf{G}_h$), with player j being the period-2 seller, and given every type profile $\mathbf{t} \in \mathbf{T}$, the number $P_{ij}(\mathbf{t}, h)$ denotes the expected value of the period-2 monetary transfer from bidder i to player j, and $Q_{ij}(\mathbf{t}, \mathbf{J})$ denotes the probability that bidder i obtains the good in period 2, with $1 - \sum_{i\neq j} Q_{ij}(\mathbf{t}, \mathbf{J})$ being the probability with which the period-2 seller keeps the good. The pair $(P_{ij}(\cdot, \mathbf{J}), Q_{ij}(\cdot, \mathbf{J}))_{i\in N\setminus\{j\}}$ is called the *period-2 outcome* given period-2 seller j and post-auction belief $\mathbf{J}$.

2.2 The equilibrium concept

An *equilibrium* consists of a bidding profile $(\beta_i(\cdot \mid h))_{i\in N, h\in\mathcal{H}_i}$, a resale-decision profile $(\gamma_h)_{h\in\mathcal{H}}$, a post-auction belief profile $(\mathbf{G}_h)_{h\in\mathcal{H}}$, and a period-2 outcome function $(P_{ij}, Q_{ij})_{i\in N,\ j\in N\setminus\{i\}}$ with the following properties:

a. for any terminal history $h \in \mathcal{H}$ where the period-1 winner is j, given the post-auction belief $\mathbf{G}_h$, there exists a Bayesian Nash equilibrium for the period-2 continuation game that induces the period-2 outcome $(P_{ij}(\cdot, \mathbf{G}_h), Q_{ij}(\cdot, \mathbf{G}_h))_{i\in N\setminus\{j\}}$;

b. for any terminal history $h \in \mathcal{H}$, if bidder i is the period-1 winner in h, then γ_h maximizes i's discounted expected payoff given the post-auction belief $\mathbf{G}_h$ and the expected period-2 outcome;

c. there exists a belief profile $\mathcal{B}$ that evolves from the common prior $(F_i)_{i\in N}$ to the post-auction belief $(\mathbf{G}_h)_{h\in\mathcal{H}}$ and obeys Bayes' rule with respect to the bidding profile $(\beta_i(\cdot \mid h))_{i\in N, h\in\mathcal{H}_i}$ and resale-decision profile $(\gamma_h)_{h\in\mathcal{H}}$ whenever possible;

d. for any bidder i, type $t_i \in T_i$, and nonterminal history $h \in \mathcal{H}_i$, the dropout price $\beta_i(t_i \mid h)$ maximizes i's discounted expected payoff given the belief profile $\mathcal{B}$ and history h, provided that everyone else abides to the bidding profile and that the resale-decision profile and period-2 outcome function are implemented.

3 Equilibrium properties in the resale game

Let $j \in N$ be the period-2 seller and $\mathbf{J}$ the post-auction belief (e.g., $\mathbf{G}_h$ in an equilibrium under terminal history h). Let J_i denote the marginal distribution induced by $\mathbf{J}$ on i's type space T_i, and $\mathbf{J}_{-i}$ the marginal distribution on $\mathbf{T}_{-i} := \prod_{k \neq i} T_k$.

Suppose a period-2 outcome $(P_{ij}(\cdot, \mathbf{J}), Q_{ij}(\cdot, \mathbf{J}))_{i \in N \setminus \{j\}}$ is implemented. Then for any period-2 bidder i ($i \neq j$) with type t_i, the probability with which i obtains the good from the period-2 seller j is equal to

$$q_{ij}(t_i, \mathbf{J}) = \int_{\mathbf{T}_{-i}} Q_{ij}(\mathbf{t}, \mathbf{J}) d\mathbf{J}_{-i}(\mathbf{t}_{-i}),$$

and the period-2 expected payoff for i is equal to

$$l_{ij}(t_i, \mathbf{J}) = t_i q_{ij}(t_i, \mathbf{J}) - \int_{\mathbf{T}_{-i}} P_{ij}(\mathbf{t}, \mathbf{J}) \mathrm{d}\mathbf{J}_{-i}(\mathbf{t}_{-i}).$$

And for any type t_j of the period-2 seller, the probability of keeping the good is equal to

$$q_j(t_j, \mathbf{J}) = 1 - \sum_{k \in N \setminus \{j\}} \int_{\mathbf{T}_{-j}} Q_{kj}(\mathbf{t}, \mathbf{J}) \mathrm{d}\mathbf{J}_{-j}(\mathbf{t}_{-j}),$$

and the period-2 expected payoff is equal to

$$w_j(t_j, \mathbf{J}) = t_j q_j(t_j, \mathbf{J}) + \sum_{k \in N \setminus \{j\}} \int_{\mathbf{T}_{-j}} P_{kj}(\mathbf{t}, \mathbf{J}) \mathrm{d}\mathbf{J}_{-j}(\mathbf{t}_{-j}).$$

By the envelope theorem, one can prove that the incentive compatibility of the period-2 outcome implies the following envelope formulas: For any $t_i, t_i' \in T_i$ and $t_j, t_j' \in T_j$,

$$l_{ij}(t_i, \mathbf{J}) - l_{ij}(t_i', \mathbf{J}) = \int_{t_i'}^{t_i} q_{ij}(x, \mathbf{J}) \mathrm{d}x, \tag{1}$$

$$w_j(t_j, \mathbf{J}) - w_j(t_j', \mathbf{J}) = \int_{t_j'}^{t_j} q_j(x, \mathbf{J}) \mathrm{d}x; \tag{2}$$

By the interim individual rationality of the period-2 outcome, we have, for any $t_i, \in T_i$

and $t_j \in T_j$,

$$t_j \ \le \ w_j(t_j, \mathbf{J}) \ \ \le \ \ \int_{\mathbf{T}_{-j}} \max\left\{t_j, \max_{k \in N \setminus \{j\}} t_k\right\} \mathrm{d}\mathbf{J}_{-j}(\mathbf{t}_{-j}), \tag{3}$$

$$0 \ \le \ l_{ij}(t_i, \mathbf{J}) \ \ \le \ \ \int_{\mathbf{T}_{-i}} \left(\max\left\{t_i, \max_{k \in N \setminus \{i\}} t_k\right\} - t_j\right) \mathrm{d}\mathbf{J}_{-i}(\mathbf{t}_{-i}). \tag{4}$$

If there are expected gains from trade between the period-2 seller and buyers, the period-2 seller j captures a nonzero share of the gains. More precisely, for any $t_j \in T_j$, if $t_j < \max T_j$ or $t_j = 0$, then, assuming resale follows any of the rules I-IV described in Section 2,

$$[(\forall i \in N \setminus \{j\}) \ \max \operatorname{support} \mathbf{J}_i > t_j] \Longrightarrow w_j(t_j, \mathbf{J}) > t_j. \tag{5}$$

Equation (5) is obviously true in the case where the period-2 seller gets to choose the resale mechanism or pick the reserve price or make an ultimatum offer. Suppose Haile's auction is used. If $t_j < \max J_i$, then the posterior distribution of j's type is nondegenerate, so the price offered by the highest buyer is greater than t_j with a positive probability unless the buyer's type is almost surely below the seller's. Hence (5) follows. If $t_j = \max J_i = 0$, with zero being the starting price in Haile's auction and the additional assumption of having more than two players in this case, the price of the good is above zero when the highest period-2 bidder gets to make an ultimatum offer to the period-2 seller, hence (5) follows. Finally, in the case where rule IV governs resale, there is a positive probability with which the period-2 seller gets to make the ultimatum offer, then (5) again follows.

3.1 A reseller sells less

Below is a new observation that plays a crucial role in the construction of our equilibria. It is aligned with the intuition that a monopolist sells less than efficiency requires: for each type of each player, the probability with which the player is the final owner of the good is higher when she is the period-2 seller than when she is a period-2 buyer.

A product distribution $\mathbf{J} := (J_i)_{i \in N}$ is called *regular* if three properties are satisfied for every $i \in N$: (i) the support of J_i is a bounded interval, (ii) J_i has a positive continuous density function J_i' on its support $\operatorname{supp} J_i$, and (iii) the *virtual utility function*, defined by

$$V_{i,\mathbf{J}}(t_i) := t_i - \frac{1 - J_i(t_i)}{J_i'(t_i)}, \qquad \forall t_i \in \operatorname{supp} J_i, \tag{6}$$

is strictly increasing on $\operatorname{supp} J_i$. Extend $V_{i,\mathbf{J}}$ beyond $\operatorname{supp} J_i$ by

$$V_{i,\mathbf{J}}(t_i) := \begin{cases} t_i & \text{if } t_i > \max \operatorname{supp} J_i \\ -\infty & \text{if } t_i < \min \operatorname{supp} J_i. \end{cases} \tag{7}$$

Lemma 1 *For any $j, k \in N$ with $j \neq k$, let $h(j), h(k) \in \mathcal{H}$ be terminal histories such that the period-1 winner is j in history $h(j)$ and is k in $h(k)$. If a post-auction belief profile $(\mathbf{G}_h)_{h\in\mathcal{H}}$ and a period-2 outcome function $(P_{ij}, Q_{ij})_{i\in N, j\in N\setminus\{i\}}$ together satisfy equilibrium condition (a), and if $\mathbf{J}$ is a regular product distribution and $\mathbf{J} = \mathbf{G}_{h(j)} = \mathbf{G}_{h(k)}$, then for any $t_k \in T_k$,*

$$q_k(t_k, \mathbf{J}) \;\; \geq \;\; q_{kj}(t_k, \mathbf{J}). \tag{8}$$

Proofs of all lemmas stated in the text are provided in Appendix A.

3.2 Comparative statics with shifts of beliefs

For any $i \in N$ and for any two alternative distribution functions J_i and H_i of i's type. We say that J_i is *squeezed upward into* H_i, denoted by $J_i \dashv H_i$, if there exists an x in the support of J_i such that H_i is equal to the posterior distribution derived from J_i conditional on the event that i's realized type $t_i \geq x$. And we say J_i is *squeezed downward into* H_i, denoted $H_i \vdash J_i$, if there exists a y in the support of J_i such that H_i is equal to the posterior distribution derived from J_i conditional on the event that i's realized type $t_i \leq y$.

For any two alternative product distributions $\mathbf{J} := (J_i)_{i\in N}$ and $\mathbf{H} := (H_i)_{i\in N}$ of the type-profile $\mathbf{t}$, we write

$$\begin{aligned} \mathbf{J} \dashv_i \mathbf{H} \;\; &\Leftrightarrow \;\; \left[J_i \dashv H_i, [\forall k \neq i, J_k = H_k]\right], \\ \mathbf{H} \vdash_i \mathbf{J} \;\; &\Leftrightarrow \;\; \left[H_i \vdash J_i, [\forall k \neq i, J_k = H_k]\right]. \end{aligned}$$

The next lemma says that a player's period-2 expected payoff is reduced if she is a period-2 buyer and the belief about her is squeezed upwards, or if she is the period-2 seller and the belief about her is squeezed downwards. The intuition behind (9) is that a period-2 seller would ask for a higher price if she believes that bidders are more likely to have high values. The intuition behind (10) is that period-2 buyers would offer lower prices if they believe that the period-2 seller is more likely to have a low use value.

Lemma 2 *Let $\mathcal{G}$ be a set of regular joint distributions on $\mathbf{T}$. Suppose that, for any $j \in N$ and $\mathbf{J} \in \mathcal{G}$, there is a nonempty set $\Omega(j, \mathbf{J})$ of period-2 outcomes, each of which is induced by some Bayesian Nash equilibria of the period-2 continuation game given period-2 seller j and post-auction belief $\mathbf{J}$. Then there exists a function $(P_{ij}, Q_{ij})_{i \in N \setminus \{j\}}$ on $\mathcal{G}$ with the following properties: (i) for each $\mathbf{J} \in \mathcal{G}$, $(P_{ij}(\cdot, \mathbf{J}), Q_{ij}(\cdot, \mathbf{J}))_{i \in N \setminus \{j\}} \in \Omega(j, \mathbf{J})$, and (ii) for any $i \neq j$, any $t_i \in T_i$, any $t_j \in T_j$, and any $\mathbf{J}, \mathbf{H} \in \mathcal{G}$,*

$$\mathbf{J} \dashv_i \mathbf{H} \quad \Rightarrow \quad l_{ij}(t_i, \mathbf{J}) \geq l_{ij}(t_i, \mathbf{H}), \tag{9}$$

$$\mathbf{H} \vdash_j \mathbf{J} \quad \Rightarrow \quad w_j(t_j, \mathbf{J}) \geq w_j(t_j, \mathbf{H}). \tag{10}$$

4 Equilibria for English auctions with resale

In this section, we construct a family of equilibria for the English auction with resale. In each equilibrium, one of the players, say player 1, is commonly expected to be the *designated winner* of the period-1 auction. Bidding strategies depend upon a *threshold* t^*, which can take on any value between 0 and $\bar{t} := \min_{i \geq 2} \bar{t}_i$. In period 1, every player bids up to her own type if it is above t^*; the designated winner with a type below t^* bids up to a certain price, to be constructed below; any other player with a type below t^* quits at zero price. If someone with type above t^* wins, resale does not occur. Otherwise, the period-1 winner with a sufficiently low type offers the good for resale in period 2 according to a continuation equilibrium analyzed in Section 3. Since the selection of the designated winner does not depend on her type and informational asymmetries remain at resale, these equilibria are inefficient, contrary to the value-bidding equilibrium of English auctions.[11]

Throughout this section we hold a threshold $t^* \in (0, \bar{t})$ fixed. Before we present the main result (Proposition 1), we state some results that are used to specify the period-1 strategy for player 1. First, we establish the existence of the period-2 continuation equilibrium (Lemma 3). This allows us to apply Lemma 2 and obtain a cutoff type at which the designated winner, player 1, upon winning in period 1, is indifferent between consuming the good and offering it for resale. Second, we show that player 1's resale decision is defined by this cutoff (Lemma 4). Third, we define prices at which each bidder $i \neq 1$ is indifferent between winning the auction and waiting for resale and show that these prices lie below the threshold t^* (Lemma 5).

Lemma 3 *For any regular joint distribution* $\mathbf{J}$ *on* $\mathbf{T}$*, given the post-auction belief* $\mathbf{J}$ *and any period-2 seller* j*, there is a Bayesian Nash equilibrium for the period-2 continuation game.*

Lemma 3 implies that the hypothesis of Lemma 2 is satisfied. Thus, there exists a selection function

$$(j, \mathbf{J}) \mapsto (P_{ij}(\cdot, \mathbf{J}), Q_{ij}(\cdot, \mathbf{J}))_{i \in N \setminus \{j\}} \tag{11}$$

such that, for any period-2 seller $j \in N$ and regular joint distribution $\mathbf{J}$ as the post-auction belief, $(P_{ij}(\cdot, \mathbf{J}), Q_{ij}(\cdot, \mathbf{J}))_{i \in N \setminus \{j\}}$ is the continuation equilibrium outcome in period 2, and the comparative statics properties (9)–(10) hold.

For any $x \in T_1$, let $\mathbf{J}_{x,t^*}$ denote the belief resulting from updating the prior distributions conditional on the event that bidder 1's type is at most x and the other bidders' types are below t^*,

$$\mathbf{J}_{x,t^*} := F_1(\cdot \mid t_1 \in [0, x]) \times \prod_{i=2}^{n} F_i(\cdot \mid t_i \in [0, t^*]).$$

If the post-auction belief is $\mathbf{J}_{t^*,t^*}$, a period-2 buyer's highest possible maximum willingness to pay is t^*, hence the sufficiently high types of bidder 1 prefer consuming the good in period 1 to offering resale in period 2, due to discounting. To find a cutoff between consumption and

[11] Haile (1999) proves that when resale after an English or second-price auction is allowed, the efficient value-bidding equilibrium remains valid.

offering resale, define

$$\tau(t^*) := \inf \{t_1 \in T_1 : t_1 \geq \delta w_1(t_1, \mathbf{J}_{t_1,t^*})\}. \tag{12}$$

Since the set on the right-hand side of (12) contains t^* and is bounded from below by zero, $\tau(t^*)$ is well-defined. The next lemma says that $\tau(t^*)$ is a cutoff such that the types of bidder 1 below it prefer offering resale to consumption, the types above it have the reverse preference, and the other bidders take the cutoff into account. Moreover, this cutoff is greater than zero, implying that the probability for the winner to offer resale is positive.

Lemma 4 *Given (11) as the period-2 continuation-equilibrium selection function, for any $t^* \in (0, \bar{t})$, we have $0 < \tau(t*) \leq t^*$ and, for any $t_1 \in T_1$,*

$$t_1 \geq \delta w_1(t, \mathbf{J}_{\tau(t^*),t^*}) \quad \textit{if} \quad t_1 > \tau(t^*), \tag{13}$$

$$t_1 \leq \delta w_1(t, \mathbf{J}_{\tau(t^*),t^*}) \quad \textit{if} \quad t_1 < \tau(t^*), \tag{14}$$

$$\tau(t^*) \longrightarrow t^* \quad \textit{as} \quad \delta \longrightarrow 1. \tag{15}$$

For all bidders $i \in N \setminus \{1\}$, let $b_i(t^*)$ denote the price that makes type t^* of bidder i indifferent between (i) winning the auction at price $b_i(t^*)$ and consuming the good, and (ii) participating only in a resale market where bidder 1 is the period-2 seller and the post-auction beliefs are $\mathbf{J}_{\tau(t^*),t^*}$:

$$b_i(t^*) \quad := \quad t^* - \delta l_{i1}(t^*, \mathbf{J}_{\tau(t^*),t^*}). \tag{16}$$

The following lemma provides bounds for $b_i(t^*)$.

Lemma 5 *For any $t^* \in (0, \bar{t})$ and $i \in N \setminus \{1\}$, we have $0 < b_i(t^*) \leq t^*$.*

Applying lemmas 2-5 the period-1 strategy for the designated winner, player 1, is as follows. If her type is above the threshold t^*, then she bids up to her type. If her type is below $\tau(t^*)$, then she bids up to $\max_{i \in S_1(h)} b_i(t^*)$, where $S_1(h) := \arg\max_{k \in N \setminus \{1\}} h_k$ denotes the set of the bidders other than 1 who remain active in the nonterminal history $h \in \mathcal{H}_1$;

and she will offer resale if she wins. If her type is between $\tau(t^*)$ and t^*, then she bids up to t^* and she will not offer resale if she wins.

Proposition 1 *For any $t^* \in (0, \bar{t})$, there exists an equilibrium for the 2-period game such that:*
(i) player 1's bidding strategy is, for any nonterminal history $h \in \mathcal{H}_1$ for player 1,

$$\beta_1(t_1 \mid h) = \begin{cases} \max_{i \in S_1(h)} b_i(t^*) & \text{if } t_1 \leq \tau(t^*) \\ t^* & \text{if } \tau(t^*) < t_1 \leq t^* \\ t_1 & \text{if } t_1 > t^*; \end{cases} \tag{17}$$

(ii) for any player $i \geq 2$, at the initial history $\mathbf{0} = (0, \ldots, 0)$ of the auction,

$$\beta_i(t_i \mid \mathbf{0}) = 0 \quad \text{if} \quad t_i \leq t^*, \tag{18}$$

and for any nonterminal history $h \in \mathcal{H}_i$,

$$\beta_i(t_i \mid h) = t_i \quad \text{if} \quad t_i > t^*; \tag{19}$$

(iii) if player 1 wins at zero price in history h, her resale decision is

$$\gamma_h(t_1) := \begin{cases} 1 & \text{if } t_1 \leq \tau(t^*) \\ 0 & \text{if } t_1 > \tau(t^*); \end{cases} \tag{20}$$

(iv) in the off-path event that a player $i \in N$ made a commonly observed deviation from the proposed period-1 strategy, the post-auction belief about t_i can be any regular distribution H_i such that

$$\begin{array}{ll} H_1 \vdash F_1 & \text{if } i = 1 \\ F_i(\cdot \mid t_i \in [0, t^*]) \dashv H_i & \text{if } i \neq 1 \text{ and } i \text{ does not offer resale} \\ F_i(\cdot \mid t_i \in [0, t^*]) = H_i & \text{if } i \neq 1 \text{ and } i \text{ offers resale}; \end{array} \tag{21}$$

(v) the period-2 outcome function is the selection function specified by (11).

The proposition is proved in Subsection 4.1. A crucial step in the proof is to show that it is not profitable for a bidder $i \geq 2$ with a type below t^* to deviate by outbidding the designated winner and attempting resale upon winning. To illustrate why such a bidder will not attempt resale, assume temporarily that there is no discounting, so that it is always optimal to offer the good for resale upon winning. The deviation switches bidder i's role from a period-2 buyer to a period-2 seller, leaving the post-auction beliefs $\mathbf{J}_{\tau(t^*),t^*}$ unchanged. By (3), $w_i(t^*, \mathbf{J}_{\tau(t^*),t^*}) = t^*$. Hence, (16) implies that type t^* is indifferent between the two roles:

$$w_i(t^*, \mathbf{J}_{\tau(t^*),t^*}) - b_i(t^*) \quad = \quad l_{i1}(t^*, \mathbf{J}_{\tau(t^*),t^*}) \quad \text{if} \;\; \delta = 1. \tag{22}$$

Rewriting both sides of the equation by the envelope formula (1)–(2) and applying the result that "a reseller sells less" (Ineq. (8)), we see that the payoff difference between type t^* and any type $t_i < t^*$ is larger in the seller role than in the buyer role:

$$w_i(t^*, \mathbf{J}_{\tau(t^*),t^*}) - w_i(t_i, \mathbf{J}_{\tau(t^*),t^*}) \quad \geq \quad l_{i1}(t^*, \mathbf{J}_{\tau(t^*),t^*}) - l_{i1}(t_i, \mathbf{J}_{\tau(t^*),t^*}).$$

This together with (22) shows that all types below t^* would rather be a period-2 buyer than a period-2 seller.

4.1 Proof of Proposition 1

By construction of the proposed equilibrium, especially its item (iv), the post-auction belief given any terminal history, on or off path, is a regular joint distribution. Thus, the period-2 outcome function (11) specified in item (v) completely describes the continuation equilibrium outcomes in period 2.

Based on the period-2 continuation equilibrium selection function, there exists a complete profile of period-1 strategies that satisfies Eqs. (17)–(20) in the proposition. Given any terminal history of the period-1 auction, there exists an optimal resale decision for the winner based on the posterior beliefs either derived from Bayes's rule on the path or specified by Eq. (21) off the path. In particular, it is easy to verify that player 1's on-path resale decision in Eq. (20) is optimal for her. For any bidder i other than player 1 and any nonterminal history $h \in \mathcal{H}_i$ where i has deviated from Eq. (18), there exists an optimal dropout price for bidder i.

The only thing left to check is that this strategy profile can be constructed in a way that satisfies Eqs. (17)–(20). We shall verify that separately for bidder 1 in Subsection 4.1.1 and any other bidder i in Subsection 4.1.2.

4.1.1 Eq. (17) constitutes a best reply for player 1

(a) For player 1, the deviant plan of losing in period 1 and trying to buy the good in period 2 is suboptimal. If $t_i \leq t^*$ for all $i \neq 1$, all other bidders quit at zero price (Eq. (18)), so if player 1 follows her deviant plan, with ties broken randomly, she loses the opportunity of winning the good at zero price and earning a nonnegative resale profit. (In the event that she wins after the tie is broken, the post-auction belief about her is squeezed downward from the prior, according to (21). Then (10), which is applicable due to the selection function (11) of continuation equilibria, implies that her continuation payoff is not larger than the one in the event that she wins without a tie.) If $t_i > t^*$ for some $i \neq 1$, any such player i bids up to t_i (Eq. (18)) and, if offering resale upon winning, would charge a resale price greater than or equal to t_i, which would not be less than the price paid by player 1 if player 1 wins in period 1.

(b) Suppose player 1's type $t_1 > t^*$. She gains no profit from winning the good and then offering resale, because either the losing bidders' use values are below t^* and hence below hers, or some of their values are above t^* and hence they bid up to their values (Eq. (18)), meaning that the price paid by player 1 has already reached the highest losing bidder's maximum willingness to pay. Coupled with the claim in (a), this implies that it is optimal for player 1 to try to win and consume the good in period 1; consequently, it is optimal for her to bid up to her value t_1, as prescribed by the equilibrium.

(c) Suppose $t_1 \leq t^*$.

(c1) It is suboptimal for player 1 to drop out at price zero, thereby losing a positive probability of winning the good at zero price and earning a nonnegative resale profit. That is because the events $t_i \leq t^*$ for all $i \neq 1$ and losing in a tie (everyone quits at zero price) both have positive probability.

(c2) Player 1 cannot gain from bidding up to a price above t^*, because her payment conditional on winning would exceed her use value and reach the other players' maximum willingness to pay (Eq. (18)).

(c3) Before the price clock starts, any positive dropout price less than or equal to t^* yields the same period-1 outcome for player 1, winning the good at zero price if $t_i \leq t^*$ for all $i \neq 1$ and losing if otherwise, hence the prescribed dropout price is a best reply.

(c4) Suppose the price clock has started and some bidder $i \neq 1$ continues bidding while no one has dropped out at a positive price. Since $t^* < \bar{t}$, player 1 believes that bidder i has a value $t_i > t^*$ and that bidder i will continue bidding up to this value. Thus player 1 believes she will not win, thus securing a nonnegative payoff, so long as she drops out before the price exceeds t^*. She cannot gain from bidding up to a price above t^*, by (c2). Hence any dropout price bounded from above by t^* is a best reply, including the proposed bidding strategy.

(c5) Suppose the price clock has started and some bidder $i \neq 1$ has dropped out at a positive price x_i. If bidder i is the last remaining rival against player 1, player 1 wins the good at the price x_i and it is too late for her to change her period-1 bidding strategy. If bidder i is not the last remaining rival, then the reasoning in (c4) applies to the remaining rival(s), so player 1 does not want to change her bidding strategy either.[12]

We have exhausted all possible nonterminal histories for bidder 1.

4.1.2 There exists a best reply for any player $i \neq 1$ that satisfies Eqs. (18)–(19)

(a) Player i has only three alternative goals:

$$\begin{aligned} X &:= \text{to win and consume the good in period 1,} \\ Y &:= \text{to buy the good in period 2,} \\ Z &:= \text{to win in period 1 and offer resale in period 2.} \end{aligned}$$

(b) Basic facts:

(b1) If player i wants X, then i bids up to his type t_i for any nonterminal history $h \in \mathcal{H}_i$.

[12] The reasoning in (c4) and (c5) is based on the fact that the deviation of a designated "loser" is not known by others unless he has dropped out at a price in $(0, t^*)$.

(b2) At the initial history $\mathbf{0}$ of the auction, if player i wants Y, then i drops out at zero price. One alternative is for player i to drop out at a bid above b_i^* and hope for a reseller other than player 1. However, such a player must have a value greater than t^* and hence she will not resell. The other alternative is for player i to drop out at a positive price below b_i^*. However, such a bid is penalized by an off-path posterior that is an upward squeeze of the distribution $F_i(\cdot \mid t_i \in [0, t^*])$ according to Eq. (21). As this deviation is unilateral, $\mathbf{J}_{\tau(t^*),t^*} \dashv_i \mathbf{H}$, then (9) implies $l_{i1}(t_i, \mathbf{J}_{\tau(t^*),t^*}) \geq l_{i1}(t_i, \mathbf{H})$, i.e., player i cannot gain from this deviation. Here (9) is applicable by the choice of the period-2 outcome function (11).

(b3) If player i wants Z, then he tops only the bid b_i^*. If he wins by topping a higher bid, then his payment is equal to either t^* (when $t_j \leq t^*$ for all $j \notin \{1, i\}$ and $\tau(t^*) < t_1 \leq t^*$, by Eq. (17)) or some $t_j > t^*$ with t_j being the highest use value among all players other than i. Player i's resale revenue cannot exceed his period-1 payment in either case.

(c) Let CE ("collusive event") denote the event "$t_j \leq t^*$ for all $j \neq i$" and let $\succeq_i$ denote player i's weak preference evaluated by i's discounted expected payoff, conditional on i's type and from the standpoint of the initial history of the auction.

(d) Conditional on CE, $t_i > t^* \Rightarrow X \succeq_i Y$ and $t_i < t^* \Rightarrow Y \succeq_i X$. The relations hold because given CE, the expected payoff from X is $t_i - b_i(t^*)$ (basic fact (b1)) and the expected payoff from Y is $\delta l_{i1}(t_i, \mathbf{J}_{\tau(t^*),t^*})$ (basic fact (b2), with the subscript $\tau(t^*)$ due to player 1's resale decision (20)). By definition (16) of $b_i(t^*)$, these two payoffs are equal to each other when $t_i = t^*$. When t_i changes from t^*, $t_i - b_i(t^*)$ changes at the rate one while $\delta l_{i1}(t_i, \mathbf{J}_{\tau(t^*),t^*})$ changes at the rate $\delta q_{i1}(t_i, \mathbf{J}_{\tau(t^*),t^*})$ (envelope formula Eq. (1)), which is less than one. Thus, the claim follows.

(e) Conditional on CE, $t_i < t^* \Rightarrow Y \succeq_i Z$. To prove that, recall basic fact (b3). If player i does achieve Z, i outbids player 1 whose type $t_1 \leq \tau(t^*)$. This fact together with (21) implies the post-auction belief is $\mathbf{J}_{\tau(t^*),t^*}$. Thus, player i's present expected payoff from achieving Z

is equal to $\delta w_i(t_i, \mathbf{J}_{\tau(t^*),t^*}) - b_i(t^*)$. By definition (16) of $b_i(t^*)$,

$$\begin{aligned} W &:= \delta w_i(t^*, \mathbf{J}_{\tau(t^*),t^*}) - b_i(t^*) \\ &\leq w_i(t^*, \mathbf{J}_{\tau(t^*),t^*}) - b_i(t^*) \\ &= t^* - b_i(t^*) \\ &= \delta l_{i1}(t^*, \mathbf{J}_{\tau(t^*),t^*}) =: L. \end{aligned}$$

By Eqs. (1)–(2), $\frac{\partial w_i}{\partial t_i}(t_i, \mathbf{J}_{\tau(t^*),t^*}) = q_i(t_i, \mathbf{J}_{\tau(t^*),t^*})$ and $\frac{\partial l_{i1}}{\partial t_i}(t_i, \mathbf{J}_{\tau(t^*),t^*}) = q_{i1}(t_i, \mathbf{J}_{\tau(t^*),t^*})$. By Lemma 1, $q_i(t_i, \mathbf{J}_{\tau(t^*),t^*}) \geq q_{i1}(t_i, \mathbf{J}_{\tau(t^*),t^*})$. Thus, the payoff from achieving Z, $\delta w_i(t_i, \mathbf{J}_{\tau(t^*),t^*}) - b_i(t^*)$, decreases from the level W faster than the expected payoff from Y, $\delta l_{i1}(t_i, \mathbf{J}_{\tau(t^*),t^*})$, decreases from the level L. As $W \leq L$, the claim is proved.

(f) Conditional on "not CE", $X \succeq_i Z$ and $Y \succeq_i Z$. That is analogous to (c2) of subsection 4.1.1.

(g) Conditional on "not CE", $t_i < t^* \Rightarrow Y \succeq_i X$ and $t_i > t^* \Rightarrow X \succeq_i Y$. That is because "not CE" implies that player i would need to pay more than t^* if he wins in period 1. If $t_i < t^*$ his payment exceeds his use value, hence $Y \succeq_i X$ follows. Mimicking the last sentence in paragraph (a) of subsection 4.1.1, we have $t_i > t^* \Rightarrow X \succeq_i Y$.

(h) If $t_i \leq t^*$, dropping out at price zero is a best reply. That is because for player i with such a type, Y is most preferred whether the event is CE or not (claims (d)–(g)). Then basic fact (b2) implies the claim, as prescribed by the equilibrium.

(i) If $t_i > t^*$, bidding up to his type t_i is a best reply. That is because for such a type of player i, X is most preferred whether the event is CE or not: If CE, $X \succeq_i Y$ by claim (d), and $X \succeq_i Z$ since all the other players' use values are below t^* and hence below player i's; if not CE, X is most preferred by claims (f)–(g). Then basic fact (b1) implies the claim.

Therefore, abiding by Eqs. (18)–(19) is part of bidder i's best reply. The proposition is hence proved.

4.2 Remarks on Proposition 1

Call the equilibrium constructed in Proposition 1, parameterized by the threshold $t^* \in (0, \bar{t})$, the t^*-*equilibrium*.

Remark 1. Proposition 1 can be extended to the case where the English auction in period 1 has a reserve price $r > 0$. To do that, slightly modify the English auction with the following amendment: The auction starts with a price level lower than r (say zero price) that corresponds to "no sale." If someone drops out at no-sale, then the price clock pauses to give others a chance to drop out. Once no more bidders drop out at no-sale, the price clock jumps to the reserve price r. Then, a t^*-equilibrium exists if $r < \bar{t}$, i.e., if the reserve price does not exceed any bidder's highest possible type. Let $t^* \in (r, \bar{t})$. Let $\hat{t} \in T_1$ be the type of bidder 1 such that her expected payoff for the entire auction-resale game is zero if she wins the good at price r and offers the good for resale, given the belief that the types in $[0, t^*]$ of other bidders participate in the resale market. Clearly $\hat{t} \leq r$. At the equilibrium, bidder 1 drops out at "no sale" if and only if her type is below $\hat{t}$. Once bidder 1 has dropped out at no-sale, other bidders play the value-bidding equilibrium; if bidder 1 does not drop out at no-sale, then the bidders' subsequent actions are analogous to the equilibria described in Proposition 1, where "dropping out at zero price" is replaced by "dropping out at no-sale." Resale occurs given the belief that (i) bidder 1's type is distributed on $[\hat{t}, \tau]$ for some $\tau \in (r, t^*)$ and (ii) the other bidders' types are distributed on $[0, t^*]$.

Remark 2. Proposition 1 can be partially extended to the case where repeated resale is allowed, i.e., after the period-2 seller has sold the good the next owner may offer resale, and so on. The t^*-equilibrium outcomes remain valid if the number of players $n = 2$, because the period-2 seller has no incentive to re-buy. Cases where $n \geq 3$ are complicated, because the dynamic nature of repeated resale requires that the continuation play during resale be predicted by perfect Bayesian equilibrium instead of the weaker Bayesian Nash equilibrium (Zheng (2002), Prop. 1). However, there is at least one nontrivial class of environments where the t^*-equilibria remain valid under repeated resale. Consider a game where any current owner of the good can commit to a mechanism that offers the good for sale, expecting that any buyer has the same option to offer resale, and so on. For such a game, Zheng (2002) constructs a perfect Bayesian equilibrium based on certain conditions for the bidder-type prior distributions; at this equilibrium, any current seller of the good chooses an auction that eventually implements the Myerson allocation from the current seller's viewpoint.

Embedded in our model, Zheng's equilibrium corresponds to a continuation (perfect Bayesian) equilibrium that captures the dynamics of repeated resale. As long as a post-auction belief satisfies Zheng's conditions on the prior distributions in his model, the continuation equilibrium induces the Myerson allocation with respect to the post-auction belief. Then one can extend properties (1)–(5), (8), and (9)–(10) and hence establish a t^*-equilibrium, provided that Zheng's conditions are satisfied by our on-path post-auction beliefs $\prod_{i=2}^n F_i(\cdot \mid t_i \in [0, t^*])$.[13] These conditions can be expressed as conditions on our prior distributions. If $n \geq 4$, these conditions are satisfied if and only if the distribution $F_3(\cdot \mid t_3 \in [0, t^*]) = \cdots = F_n(\cdot \mid t_n \in [0, t^*])$ has a weakly decreasing density and dominates $F_2(\cdot \mid t_2 \in [0, t^*])$ in terms of the hazard rate.[14]

Remark 3. The t^*-equilibrium construction makes essential use of the transparent dynamic nature of an English auction, because the designated winner's dropout price depends on the set of the other bidders who have not dropped out (the upper branch of Eq. (17)). This dependence is important in our construction because bidders drawn from different distributions need different prices $b_i(t^*)$ to be kept obedient to the threshold t^* (Eq. (16)). For exactly this reason, the t^*-equilibrium construction does not generally extend to second-price auctions except for the case where bidders 2 to n are ex-ante symmetric, implying $b_2(t^*) = \cdots = b_n(t^*)$.

Remark 4. Equilibria with small t^* are more robust than equilibria with large t^* in the sense that, at a t^*-equilibrium, common knowledge of prior probability distributions is

[13]The reason is that, in establishing a t^*-equilibrium, we need only to consider $F_i(\cdot \mid t_i \in [0, t^*])$ as the posterior distribution for any player i. This is the posterior distribution for any i on the equilibrium path given that resale is offered. For off-path events of the period-1 auction, the post-auction beliefs are specified by (21). In the first branch of (21), the only post-auction belief affected is that of player 1, who is still the period-2 seller in that off-path event. This change is immaterial because the belief about a seller does not affect the Myerson allocation. In the second branch of (21), the post-auction belief of every player j remains to be $F_j(\cdot \mid t_j \in [0, t^*])$ except the deviant losing bidder i. Let "$t_i = t^*$" be the post-auction belief of i, then the resale price offered to i is at least t^*, making it unprofitable for i to deviate (hence step (b2) in subsection 4.1.2 follows). The third branch of (21) does not alter anyone's post-auction belief.

[14]See Mylovanov and Tröger (2006, Corollary 1). Their paper also characterizes the weaker conditions in the case of $n = 3$, which corresponds to the 2-bidder case in Zheng (2002).

required only on the interval $[0, t^*]$.

Remark 5. The t^*-equilibria are not the only equilibria that differ from the value-bidding equilibrium. There also exist "extreme equilibria" where bidder 1's planned drop-out price is so high that, for every bidder $i \geq 2$, even the highest type $\bar{t}_i$ finds it optimal to drop out at the beginning of the auction.[15] Extreme equilibria require common knowledge of the entire prior distributions, for the resale continuation game. Moreover, there are practical reasons why extreme equilibria might not be played. First, in an extreme equilibrium, the good is always sold at zero price at the initial auction. That would make a regulator suspicious of bidding collusion, which the bidders may want to avoid. Second, if low-type bidders have a budget constraint that prevents them from bidding up to the highest possible type, a designated winner's bidding strategy in an extreme equilibrium is not credible. (Brusco and Lopomo (2006) made this point previously in a no-resale model.) A t^*-equilibrium, by contrast, can survive a budget constraint if t^* is less than the budget.

5 The interim Pareto dominance of collusion

By randomizing over the choice of the designated winner, any t^*-equilibrium in Proposition 1 can be transformed into a *t^*-collusive equilibrium* for the auction-resale game which, from the viewpoint of the bidders in the initial auction, interim Pareto-dominates the value-bidding equilibrium for a nondegenerate range of t^* (Proposition 2). A public randomization device, or sunspot (see Shell, 1977 and Cass and Shell, 1989), may be used. As in the case of correlated equilibrium, once the randomization is complete (i.e., the designated winner is determined) it is in the best interest of all players to play their assigned roles, given their belief that others will do so. Consequently, collusive bidding is self-enforced as an equilibrium without relying on any repeated-game setup, pre-auction inter-bidder transfers, or post-auction (possibly illegal) enforcement to divide the spoil.

A t^*-collusive equilibrium is collusive in its nature because the sunspot assignment of roles among the bidders is a form of tacit collusion and because the equilibrium makes every

[15] Zheng (2000, Section 5.2) constructs an extreme equilibrium in a second-price-auction-type mechanism with reserve prices. See also Garratt and Tröger (2006).

type of every bidder better-off than the (socially) efficient value-bidding equilibrium. The possibility of resale plays a pivotal role in this construction. In the same environment where the interim bidder-Pareto superiority of collusion is established, we shall show that, should resale be banned, there are always some bidder types who strictly prefer the value-bidding equilibrium to the collusive equilibrium (Proposition 3).

Throughout this section, we assume that there is a sunspot which has n possible states, each with probability $1/n$, and whose realization, commonly observed, takes place after the bidders have been privately informed and before the period-1 auction starts. For any $t^* \in (0, \bar{t})$, a *t^*-collusive equilibrium* is: Enumerate the possible sunspot states by the names of the bidders; if the realized state is j, then player j is the designated winner, and the t^*-equilibrium, with player j taking the role of player 1 in Proposition 1, is played. Clearly this constitutes an equilibrium.

Next we establish the interim bidder-Pareto superiority of a t^*-collusive equilibrium for all sufficiently small t^* in a symmetric, no-discounting environment.

Proposition 2 *If resale is governed by rule I defined in Section 2, if*

$$F_1 = \cdots = F_n, \quad \delta = 1, \tag{23}$$

and if $t^* > 0$ is sufficiently close to 0, then every type of every bidder is strictly better-off in a t^*-collusive equilibrium than in an equilibrium where everybody bids their value.

The proof is in Appendix C. To provide some intuition for Proposition 2 we now show that under resale rule I, *if each bidder's type is uniformly distributed on $[0, 1]$, with no discounting, then for any $t^* \in (0, 1)$ the t^*-collusive equilibrium interim Pareto dominates (for the bidders) the value-bidding equilibrium.* This illustrates the essential idea behind the proof of Proposition 2, which is to use a linearization, Eq. (39).

First we provide a lemma, which is also useful for the proof of Proposition 2.

Lemma 6 *Suppose that condition (23) holds. If type t^* is better-off (resp. strictly better-off) in the t^*-collusive equilibrium, then all types above t^* are also better-off (resp. strictly better-off).*

To evaluate the uniform distribution case, pick any $t \in [0, t^*]$. In the value-bidding equilibrium, a type-t bidder's expected payoff is

$$U^{\text{val}}(t) = \int_0^t F(x)^{n-1} dx = \int_0^t x^{n-1} dx = \frac{t^n}{n}.^{16}$$

For any $x \in [0, t^*]$, let

$$V_*(x) := x - \frac{F(t^*) - F(x)}{f(x)} = 2x - t^*, \quad \text{so} \quad V_*^{-1}(x) = \frac{x + t^*}{2}.$$

In the t^*-collusive equilibrium, a bidder's payoff depends on whether or not she is selected as the designated winner. As the resale mechanism is assumed to be chosen by the period-2 seller, the period-2 outcome is the Myerson allocation according to V_*. For any $x \in [0, t^*]$, the probability with which a type-x designated winner is the final owner of the good is equal to

$$F(V_*^{-1}(x))^{n-1}.$$

Thus, by the envelope theorem, in the t^*-collusive equilibrium, the expected payoff for a designated winner of type $t \in [0, t^*]$ is

$$\begin{aligned}
U_w(t) &= t^* F(t^*)^{n-1} - \int_t^{t^*} F(V_*^{-1}(x))^{n-1} dx \\
&= t^{*n} - \int_t^{t^*} \left(\frac{x + t^*}{2}\right)^{n-1} dx \\
&= t^{*n} - \int_{\frac{t+t^*}{2}}^{t^*} 2y^{n-1} dy \\
&= t^{*n} - \frac{2}{n}\left(t^{*n} - \left(\frac{t + t^*}{2}\right)^n\right),
\end{aligned}$$

and the expected payoff $U_l(t)$ for a designated loser of type $t \in [0, t^*]$ is nonnegative by individual rationality in the resale continuation game,

$$U_l(t) \geq 0.$$

[16]The derivation of this expected payoff and ones that follow are explained in more detail in the proof of Proposition 2.

A type-t bidder's expected payoff in the t^*-collusive equilibrium is

$$U^{\mathrm{col}}(t) = \frac{1}{n}U_w(t) + \frac{n-1}{n}U_l(t),$$

and the bidder's net gain from the t^*-collusive equilibrium relative to the value-bidding equilibrium is

$$\begin{aligned}
\Delta(t) &:= U^{\mathrm{col}}(t) - U^{\mathrm{val}}(t) \\
&= \frac{1}{n}\left[t^{*n} - \frac{2}{n}\left(t^{*n} - \left(\frac{t+t^*}{2}\right)^n\right)\right] + \frac{n-1}{n}U_l(t) - \frac{1}{n}t^n \\
&\geq \frac{1}{n}\left[t^{*n} - \frac{2}{n}\left(t^{*n} - \left(\frac{t+t^*}{2}\right)^n\right)\right] - \frac{1}{n}t^n \quad \text{since } U_l(t) \geq 0.
\end{aligned}$$

Let

$$y := \frac{t}{t^*}.$$

Since $t \in [0, t^*]$, $y \leq 1$ and $n \geq 2$,

$$\begin{aligned}
\Delta(t) &\geq \frac{t^*}{n}\left[1 - \frac{2}{n} + \frac{2}{n}\left(\frac{y+1}{2}\right)^n - y^n\right] \\
&\geq \frac{t^*}{n}\left[1 - \frac{2}{n} + \frac{2}{n}y^n - y^n\right] \\
&= \frac{t^*}{n}\left(1 - \frac{2}{n}\right)(1 - y^n) \\
&\geq 0.
\end{aligned}$$

This fact, combined with Lemma 6, establishes the desired result.

Below we provide some remarks on Proposition 2:

Remark 1. The collusion result extends to environments with a discount factor sufficiently close to 1; this follows from Proposition 1 and Eq. (15).

Remark 2. The collusion result extends to environments with asymmetric bidders that are approximately symmetric in an appropriate sense; this follows from continuity and Proposition 1. By using a non-uniformly distributed sunspot variable it may be possible to extend the collusion result to a larger class of asymmetric environments. In some very asymmetric environments, however, collusion is not possible if a small t^* is used. Consider environments with at least two private-value bidders and a pure speculator as in Garratt and Tröger (2006): if t^* is small, then the initial seller's revenue in a t^*-equilibrium is larger than in the value-bidding equilibrium, which means that the bidders' aggregate payoff is smaller (Garratt and Tröger, 2006, online supplement, Proposition 5).

Remark 3. The uniform example shows that the gains to playing a collusive equilibrium can be quite large. Table 1 shows the gains to a bidder with type $t^* = .9$ in an environment with F uniform on $[0, 1]$, for various numbers of participants. The gains to type t^* are the minimum gains over all types in this example.

n	$U^{\text{val}}(.9)$	$U^{\text{col}}(.9)$	% increase
2	0.405	0.50625	25
5	0.1181	0.19043	61.24
10	0.03487	0.06277	80.01

Table 1: $F(t) = t$, $t^* = .9$

Remark 4. The collusion result extends to an English auction with a small reserve price. This follows from Remark 1 after Proposition 1, by continuity. For larger reserve prices r, the question is whether bidders can collude so that the interim expected payoff of any bidder-type above r is larger than in the value-bidding equilibrium with reserve price r (where bidders with use values below the reserve price abstain). We have three results for environments where the type distribution F is uniform. First, if $n = 2$, then such an interim Pareto-dominating equilibrium exists for any reserve price below 1.[17] Second, if $n \geq 4$, then an equilibrium that interim Pareto-dominates value-bidding exists if the optimal reserve price under value-bidding, 1/2, is used. Third, an interim Pareto-dominating equilibrium

[17] We suspect the result extends to $n \geq 3$, but the computations would be more complex.

exists for any reserve price arbitrarily close to 1 if types are uniformly distributed on $[0, 1]$ and the number of bidders is sufficiently large.

Remark 5. The collusion result goes through if resale is conducted using a standard auction with optimal reserve price (Rule II). This is true because we assume ex-ante symmetric bidders in Proposition 2 and hence, the outcome is identical to that of the Myerson optimal auction.

5.1 The importance of resale

Equilibria with a bidding structure similar to our t^*-equilibria exist in the model that bans resale (Blume and Heidhues (2004)). There, bidders with use value above t^* bid their use values, all bidders except a special bidder bid 0 if their use values are below t^*, and the special bidder bids t^* if her use value is below t^*. Based on the apparent similarities to our construction, one might conjecture that, if a public randomization device is used to determine the designated high bidder, then one can obtain a collusive equilibrium in the no-resale model that interim Pareto dominates the value-bidding equilibrium, as in our resale model. However, for a large class of value distributions including all strictly concave distributions this is not the case. Without the possibility of resale, a high-value bidder gains too little from collusion.

For any $t^* \in (0, 1]$, call the counterpart of our t^*-collusive equilibrium in the no-resale model the *t^*-collusive no-resale equilibrium.*

Proposition 3 *Suppose resale is not permitted, that condition (23) holds with the identical distribution denoted by F, and F is strictly concave. Then for any $t^* \in (0, 1]$, there exits a type $t' < t^*$ such that any bidder of type $t > t'$ strictly prefers the value-bidding equilibrium to the t^*-collusive no-resale equilibrium.*

Proof. In a t^*-collusive no-resale equilibrium, where each bidder is selected as the special bidder with equal probability, the expected payoff for a type-t^* bidder equals

$$U^*(t^*) \quad = \quad F(t^*)^{n-1} \frac{1}{n} t^*.$$

The payoff of type t^* in the value-bidding equilibrium equals

$$U^{\mathrm{val}}(t^*) \quad = \quad \int_0^{t^*} F(t)^{n-1}\mathrm{d}t.$$

Strict concavity of F implies

$$\forall 0 < t < t^* : \frac{F(t)}{t} > \frac{F(t^*)}{t^*}.$$

Therefore,

$$U^{\mathrm{val}}(t^*) > \frac{F(t^*)^{n-1}}{(t^*)^{n-1}} \int_0^{t^*} t^{n-1}\mathrm{d}t = \frac{F(t^*)^{n-1}}{(t^*)^{n-1}} \frac{1}{n}(t^*)^n = U^*(t^*). \tag{24}$$

By the continuity of U^* and U^{val}, Ineq. (24) implies that there exists a $t' \in (0, t^*)$ such that $U^{\mathrm{val}}(t) - U^*(t) > 0$ for all $t \in (t', t^*]$. That establishes the preference in favor of the value bidding equilibrium for all types $t \in (t', t^*]$.

Now consider a bidder of type $t > t^*$. A bidder with type $t > t^*$ has a different payoff in the t^*-collusive no-resale equilibrium than under value bidding only if the highest value among everyone else, call it y, is less than t^*. In this case, under collusion, he gets the good for price zero with probability $1/n$ and he gets it for price t^* with probability $1 - 1/n$. In contrast, under value bidding, he gets the good for sure (in the event that $y \leq t^*$) at a price equal to y. He prefers value biding if the expected value of y, conditional on the event that $y \leq t^*$, is less than $(1 - 1/n)t^*$. This is true since

$$\begin{aligned}
\mathbb{E}[y|y < t^*] \quad &= \quad t^* - \frac{1}{F(t^*)^{n-1}} \int_0^{t^*} F(y)^{n-1}\mathrm{d}y \\
&= \quad t^* - \frac{1}{F(t^*)^{n-1}} \int_0^{t^*} y^{n-1} \left(\frac{F(y)}{y}\right)^{n-1} \mathrm{d}y \\
&< \quad t^* - \frac{1}{(t^*)^{n-1}} \int_0^{t^*} y^{n-1}\mathrm{d}y \\
&= \quad t^* - \frac{t^*}{n} \\
&= \quad (1 - 1/n)t^*,
\end{aligned}$$

where the inequality follows from the strict concavity of F. ■

A The proofs of Lemmas 1-6

Proof of Lemma 1. We consider each of the four rules for resale described in Section 2.

Rule I: The period-2 seller picks the resale mechanism. Then Myerson's (1981) characterization of optimal auctions is applicable.[18] By the regular assumption of $\mathbf{J}$, the resale mechanism is the Myerson auction based on the virtual utility functions $V_{i,\mathbf{J}}$ defined by (6)–(7), which awards the good in descending order of $V_{i,\mathbf{J}}(t_i)$ down to the period-2 seller's type. Then "player k gets the good given history $h(j)$" implies

$$V_{k,\mathbf{J}}(t_k) \geq \max\{\max_{i \neq j,k} V_{i,\mathbf{J}}(t_i), t_j\},$$

which implies $t_k > \max_{i \neq k} V_{i,\mathbf{J}}(t_i)$, which in turn implies "player k keeps the good given history $h(k)$."

Rule II: The resale mechanism is an English or second-price auction with a uniform reserve price r chosen by the period-2 seller. Here "player k gets the good given history $h(j)$" implies $t_k \geq \max\{\max_{i \neq j,k} t_i, r\}$, which, since r is greater than or equal to a period-2 seller's type, implies $t_k \geq t_i \forall i \neq k$, hence player k keeps the good given history $h(k)$.

Rule III: The resale mechanism is Haile's auction. Then the reasoning is analogous to that for rule II, as a period-2 seller rejects all the bids if they are below her type.

Rule IV: Player-specific bargaining power in the 2-bidder environment. If the period-2 seller is the resale-offer proposer, the reasoning is the 2-player special case of rule I. If the period-2 buyer is the proposer, the reasoning is the 2-player special case of rule II. ■

Proof of Lemma 2. We consider each of the four rules for resale described in Section 2.

Rule I: As in the proof of Lemma 1, under this rule, the resale mechanism in any period-2 continuation equilibrium is the Myerson auction according to the virtual utility functions defined by (6)–(7) based on the post-auction belief. Property (10) holds with equality ($w_j(t_j, \mathbf{J}) = w_j(t_j, \mathbf{H})$) because in the Myerson auction, the belief about the period-2 seller

[18] In contrast to Myerson's assumptions, we allow for the possibility that the period-2 seller is privately informed about her type. As shown in Milovanov and Tröger (2007), the resulting informed-principal game has an equilibrium such that the period-2 seller offers the same resale mechanism as when her type is publicly known, as assumed by Myerson.

has no effect on bidders' actions. To verify (9), let $\mathbf{J} \dashv_i \mathbf{H}$, i.e., $J_i \dashv H_i$ and $J_k = H_k$ for all $k \neq i$. Then $V_{k,\mathbf{J}} = V_{k,\mathbf{H}}$ for all $k \neq i$, and $V_{i,\mathbf{J}} \geq V_{i,\mathbf{H}}$ by the definition of upward squeezing and Eqs. (6)–(7). Thus, for all $t_i \in T_i$,

$$q_{ij}^{\text{Myerson}}(t_i, \mathbf{H}) \quad \leq \quad q_{ij}^{\text{Myerson}}(t_i, \mathbf{J}).$$

Property (9) follows from (1) together with the fact that in a seller-optimal mechanism, the expected payoff for the lowest type of bidder i is zero.

Rule II: The resale mechanism is an English or second-price auction given a reserve price r chosen by the period-2 seller. Here property (10) holds with equality ($w_j(t_j, \mathbf{J}) = w_j(t_j, \mathbf{H})$) because the belief about the period-2 seller has no impact on bidding behavior. Property (9) follows from Proposition 4 in Appendix B, which says that, given the existence of the equilibria for the period-2 continuation game, there is a monotone selection such that an upwards squeeze of the belief about a bidder makes the period-2 seller either increase the reserve price or leave it unchanged.

Rule III: The resale mechanism is an English auction without reserve and with an option for the period-2 seller to reject all the bids after the bidding process has stopped (Haile (2003)). Given any post-auction belief, the continuation game has an equilibrium: the period-2 seller sells the good at the highest bid if it exceeds her type and rejects all bids if the highest bid is below her type; every period-2 bidder stays in until the current price reaches his type and, once he is the only bidder left in the auction, bids up to an ultimatum offer, optimally chosen based on his belief about the seller's type. At this equilibrium, the belief about a bidder's type does not affect the bidding behavior, hence (9) holds with equality ($l_{ij}(t_i, \mathbf{J}) \geq l_{ij}(t_i, \mathbf{H})$). Property (10) holds because a downward squeeze of the belief about the period-2 seller's type implies period-2 buyers think that the seller is more willing to accept low prices than without the squeeze, so they offer the seller lower prices.

Rule IV: Player-specific bargaining power in the 2-bidder case. If the player who gets to propose in period 2 is the period-2 seller, then we are back to a special case of rule I. If the proposer in period-2 is the period-2 buyer, then we are back to a special case of rule III where the only remaining period-2 bidder makes an ultimatum offer to the period-2 seller.

Hence both properties (9) and (10) hold. ■

Proof of Lemma 3. The existence of equilibrium under resale rule I is well known. Under rule II, given any reserve price, value-bidding constitutes an equilibrium. The equilibrium under resale rule III has been sketched in the proof of Lemma 2. (Haile (2003) has details.) Rule IV is a special case of either rule I, if the period-2 seller gets to be the proposer, or rule III, if the period-2 buyer proposes. ■

Proof of Lemma 4. By definition of $\tau(t^*)$, $\tau(t^*) \le t^*$. To prove that $\tau(t^*) > 0$, observe that $w_1(0, \mathbf{J}_{0,t^*}) > 0$ by (5). Hence $\hat{t} := \delta w_1(0, \mathbf{J}_{0,t^*}) > 0$. Thus, for all $t < \hat{t}$,

$$t < \delta w_1(0, \mathbf{J}_{0,t^*}) \overset{(2)}{\le} \delta w_1(t, \mathbf{J}_{0,t^*}) \overset{(10)}{\le} \delta w_1(t, \mathbf{J}_{t,t^*}),$$

implying $\tau(t^*) \ge \hat{t} > 0$. Here (10) holds due to the continuation-equilibrium selection function (11).

To prove (13), let $t > \tau(t^*)$. By Eq. (12), there exists $v \in (\tau(t^*), t)$ such that

$$v \quad \ge \quad \delta w_1(v, \mathbf{J}_{\tau(t^*),t^*}). \tag{25}$$

From (2) and $q_1(s, \mathbf{J}_{\tau(t^*),t^*}) \le 1$,

$$w_1(t, \mathbf{J}_{\tau(t^*),t^*}) - w_1(v, \mathbf{J}_{\tau(t^*),t^*}) \quad \le \quad t - v. \tag{26}$$

Putting (25) and (26) together we obtain $t \ge \delta w_1(t, \mathbf{J}_{\tau(t^*),t^*})$.

To prove (14), we first claim that

$$\tau(t^*) \le \delta w_1(\tau(t^*), \mathbf{J}_{\tau(t^*),t^*}). \tag{27}$$

Suppose not. Then, by the proved fact $\tau(t^*) > 0$ and the continuity of $w_1(\cdot, \mathbf{J}_{\tau(t^*),t^*})$, there exists a $t \in (0, \tau(t^*))$ close enough to $\tau(t^*)$ such that

$$t > \delta w_1(t, \mathbf{J}_{\tau(t^*),t^*}) \overset{(10)}{\ge} \delta w_1(t, \mathbf{J}_{t,t^*}),$$

contradicting the definition of $\tau(t^*)$. Hence Ineq. (27) holds. Thus, if we define

$$\phi(t) := t - \delta w_1(t, \mathbf{J}_{\tau(t^*),t^*}),$$

we know that $\varphi(\tau(t^*)) \leq 0$. Furthermore, by (2), the derivative

$$\phi'(t) = 1 - \delta q_1(t, \mathbf{J}_{\tau(t^*),t^*}) \geq 0.$$

Thus, $\phi(t) \leq 0$ for all $t \in [0, \tau(t^*))$, i.e., (14) is true.

To prove the limit result (15), let $\epsilon > 0$ and define

$$\overline{\delta} \;:=\; \sup_{t \leq t^* - \epsilon} \frac{t}{w_1(t, \mathbf{J}_{0,t^*})}.$$

By (2), $w_1(t, \mathbf{J}_{0,t^*})$ is continuous in t. Hence, (5) implies that $\overline{\delta} < 1$. For all $\delta > \overline{\delta}$ and $t \leq t^* - \epsilon$,

$$\delta w_1(t, \mathbf{J}_{t,t^*}) \overset{(10)}{\geq} \delta w_1(t, \mathbf{J}_{0,t^*}) > \overline{\delta} w_1(t, \mathbf{J}_{0,t^*}) \geq t.$$

Then, by the definition of $\tau(t^*)$, we have $\tau(t^*) \geq t^* - \epsilon$. ■

Proof of Lemma 5. By Lemma 4, $\tau(t^*) > 0$. Hence (4) implies $l_{i1}(t^*, \mathbf{J}_{\tau(t^*),t^*}) < t^*$, so (16) implies $0 < b_i(t^*)$. The inequality $b_i(t^*) \leq t^*$ follows from the lower bound 0 in (4). ■

Proof of Lemma 6. Let $t \geq t^*$. For any player i with type t, the only way his expected payoffs are different between the t^*-collusive-equilibrium and the value-bidding equilibrium is if the highest type $t_{-i}^{(1)}$ of his rivals is below t^*. When this occurs, since $t \geq t^*$, bidder i is the final owner of the good in either equilibrium, hence the bidder's payoff-difference between the two equilibria is equal to the difference in expected payments in each case. This payment-difference is the same for all $t \geq t^*$: in the t^*-collusive-equilibrium, bidder i pays zero with probability $1/n$ and pays $b_i(t^*)$ with probability $(n-1)/n$; in the value-bidding equilibrium, bidder i pays $t_{-i}^{(1)}$. ■

B Optimal reserve prices in English or second-price auctions

Let us consider the following game: A seller, player 0, is to offer an item for sale through an English or second-price auction with a reserve price r uniform for all bidders; there are n bidders, named $1, \ldots, n$; the use-value ("type") t_i of the item for bidder i is independently drawn from a commonly known distribution F_i, with positive density f_i on its support $[\underline{t}_i, \bar{t}_i]$. Assume that the virtual utility function $V_i(t_i) := t_i - (1 - F_i(t_i))/f_i(t_i)$ is well-defined and strictly increasing on $[\underline{t}_i, \bar{t}_i]$ for each bidder i.

Only the value-bidding equilibrium will be considered.

For each bidder i, let $F_{-i}^{(1)}$ denote the distribution function for $\max\{t_j : j \neq i\}$. Given the reserve price r and the value-bidding equilibrium, the probability with which a type-t_i bidder i wins is equal to $F_{-i}^{(1)}(t_i)\mathbf{1}_{t_i \geq r}$. By the envelope theorem, the surplus for this bidder is equal to

$$U_i(t_i) = (\mathbf{1}_{t_i \geq r}) \int_r^{t_i} F_{-i}^{(1)}(x)dx \tag{28}$$

and the expected payment delivered by this bidder is equal to

$$p_i(t_i) = (\mathbf{1}_{t_i \geq r}) \left(t_i F_{-i}^{(1)}(t_i) - \int_r^{t_i} F_{-i}^{(1)}(x)dx \right).$$

At this equilibrium, the seller's surplus extracted from bidder i is equal to

$$\begin{aligned} \mathbb{E}_{t_i}\left[p_i(t_i) - t_0 F_{-i}^{(1)}(t_i)\mathbf{1}_{t_i \geq r}\right] &= \mathbb{E}_{t_i}\left[(\mathbf{1}_{t_i \geq r}) F_{-i}^{(1)}(t_i)(t_i - t_0)\right] \\ &\quad -\mathbb{E}_{t_i}\left[(\mathbf{1}_{t_i \geq r}) \int_r^{t_i} F_{-i}^{(1)}(x)dx\right]. \end{aligned} \tag{29}$$

If $r < \underline{t}_i$, then $\int_r^{\underline{t}_i} F_{-i}^{(1)}(x)dx = 0$, otherwise $F_{-i}^{(1)} > 0$ on a positive-measure subset of $[r, \underline{t}_i]$, and consequently the seller is better-off by raising the reserve price to $\underline{t}_i$, which generates the same probability of sale (since bidder i bids at least $\underline{t}_i$) and increases the expected revenue conditional on a sale (since the highest type of i's rivals is probably below $\underline{t}_i$). Thus, we may assume, without loss of generality due to (28) and (29), that

$$r \geq \underline{t}_i \quad \forall i = 1, \ldots, n. \tag{30}$$

By (30), the third expected value in Eq. (29) is equal to

$$\begin{aligned}\left(\mathbf{1}_{\bar{t}_i \geq r}\right) \int_r^{\bar{t}_i} \int_r^{t_i} F_{-i}^{(1)}(x) f_i(t_i) dx dt_i &= \left(\mathbf{1}_{\bar{t}_i \geq r}\right) \int_r^{\bar{t}_i} \int_x^{\bar{t}_i} F_{-i}^{(1)}(x) f_i(t_i) dt_i dx \\ &= \mathbb{E}_{t_i}\left[\frac{1 - F_i(t_i)}{f_i(t_i)} \left(\mathbf{1}_{t_i \geq r}\right) F_{-i}^{(1)}(t_i)\right].\end{aligned}$$

Then, by Eq. (29), the seller's surplus extracted from bidder i is equal to

$$\mathbb{E}_{t_i}\left[\left(V_i(t_i) - t_0\right)\left(\mathbf{1}_{t_i \geq r}\right) F_{-i}^{(1)}(t_i)\right].$$

Thus, the total surplus for the seller, extracted from all the bidders, equals

$$\begin{aligned}\sum_{i=1}^n \mathbb{E}_{t_i}\left[\left(V_i(t_i) - t_0\right)\left(\mathbf{1}_{t_i \geq r}\right) F_{-i}^{(1)}(t_i)\right] &= \sum_{i=1}^n \int_r^{\max\left\{\bar{t}_i, r\right\}} F_{-i}^{(1)}(t_i) f_i(t_i) \left(V_i(t_i) - t_0\right) dt_i \\ &= \sum_{i=1}^n \int_r^{\bar{t}_i} F_{-i}^{(1)}(t_i) f_i(t_i) \left(V_i(t_i) - t_0\right) dt_i,\end{aligned}$$

where the second equality follows from the fact that $f_i(t_i) = 0$ if $r > t_i > \bar{t}_i$. (Extend V_i by $V_i(x) := x$ for all $x > \bar{t}_i$.) Therefore, an optimal reserve price r^* maximizes

$$\begin{aligned}\pi\left(r \mid (\underline{t}_i)_{i=1}^n\right) &:= \sum_{i=1}^n \int_r^{\bar{t}_i} F_{-i}^{(1)}(t_i) f_i(t_i) \left(V_i(t_i) - t_0\right) dt_i \\ &= \sum_{i=1}^n \int_r^{\bar{t}_i} \left(\prod_{j \neq i} F_j(t_i)\right) f_i(t_i) \left(V_i(t_i) - t_0\right) dt_i \end{aligned} \tag{31}$$

over all $r \geq \max\left\{t_0, \max_{i=1,\ldots,n} \underline{t}_i\right\}$. The notation $\pi\left(\cdot \mid (\underline{t}_i)_{i=1}^n\right)$ signifies the possible dependence on the parameters $(\underline{t}_i)_{i=1}^n$.

The question that emerged with regards to Rule II in the proof of Lemma 2 is how the seller's optimal reserve price r^* varies with an upward squeeze of F_i, i.e., an increase in $\underline{t}_i$ (without upsetting $\underline{t}_i \leq \bar{t}_i$).

Lemma 7 *If r^* is an optimal reserve price given $(\underline{t}_i)_{i=1}^n$ $(:= (\underline{t}_k, \underline{t}_{-k})$ for any bidder k) and if $\underline{t}_k < \underline{t}'_k \leq r^*$, then r^* is an optimal reserve price given $(\underline{t}'_k, \underline{t}_{-k})$.*

Proof. With $\underline{t}_k$ replaced by $\underline{t}'_k \in (\underline{t}_k, \bar{t}_k]$, $F_k(\cdot)$ and $f_k(\cdot)$ are replaced by $F_k(\cdot)/F_k(\underline{t}'_k)$ and $f_k(\cdot)/F_k(\underline{t}'_k)$, and V_k is unchanged on $[\underline{t}'_k, \bar{t}_k]$. Thus, by Eq. (31), for any $r \geq \max\{\underline{t}'_k, \max_{i\neq k} \underline{t}_k\}$,

$$\begin{aligned}
\pi(r \mid \underline{t}'_k, \underline{t}_{-k}) &= \sum_{i\neq k} \int_r^{\bar{t}_i} \frac{F_k(t_i)}{F_k(\underline{t}'_k)} \left(\prod_{j\neq k,i} F_j(t_i) \right) f_i(t_i) \left(V_i(t_i) - t_0\right) dt_i \\
&\quad + \int_r^{\bar{t}_k} \left(\prod_{j\neq k} F_j(t_k) \right) \frac{f_k(t_k)}{F_k(\underline{t}'_k)} \left(V_k(t_k) - t_0\right) dt_k \\
&= \frac{1}{F_k(\underline{t}'_k)} \pi(r \mid \underline{t}_k, \underline{t}_{-k}) \quad \text{by factoring out } F_k(\underline{t}'_k).
\end{aligned}$$

Since r^* maximizes $\pi(r \mid \underline{t}_k, \underline{t}_{-k})$ over all $r \geq \max\{t_0, \max_{i=1,\ldots,n} \underline{t}_i\}$ and $\max_{i=1,\ldots,n} \underline{t}_i \leq \max\{\underline{t}'_k, \max_{i\neq k} \underline{t}_i\}$, r^* also maximizes $\pi(r \mid \underline{t}'_k, \underline{t}_{-k})$ over all $r \geq \max\{t_0, \max\{\underline{t}'_k, \max_{i\neq k} \underline{t}_i\}\}$. Any $r < \max\{t_0, \max\{\underline{t}'_k, \max_{i\neq k} \underline{t}_i\}\}$ is ruled out by the boundary condition (30). ■

Lemma 8 *If r^* is an optimal reserve price given $(\underline{t}_k, \underline{t}_{-k})$ and if $\underline{t}_k$ increases to any $\underline{t}'_k$ such that $\bar{t}_k \geq \underline{t}'_k > r^*$, then any optimal reserve price given $(\underline{t}'_k, \underline{t}_{-k})$ is greater than r^*.*

Proof. This follows directly from the boundary condition (30). ■

Proposition 4 *For any bidder k, there is a monotone selection $\underline{t}_k \mapsto r^*(\underline{t}_k)$, with $r^*(\underline{t}_k)$ being an optimal reserve price given $(\underline{t}_k, \underline{t}_{-k})$, such that for all $\underline{t}_k \leq \bar{t}_k$, (i) $r^*(\underline{t}_k)$ weakly increases in $\underline{t}_k$ and (ii) the surplus $U_j(t_j)$ of any bidder j of any type t_j, given the reserve price $r^*(\underline{t}_k)$, weakly decreases in $\underline{t}_k$.*

Proof. Claim (i) follows directly from the previous lemmas. Claim (ii) follows from claim (i) and Eq. (28). ■

The choice of an optimal reserve price, uniform for all bidders, is different from the counterpart in the Myerson auction, where the seller can discriminate across bidders. To illustrate the difference, consider a 2-bidder case such that the optimal reserve r^* is an interior solution. With only two bidders, the first-order necessary condition is

$$F_2(r^*) f_1(r^*)(t_0 - V_1(r^*)) + F_1(r^*) f_2(r^*)(t_0 - V_2(r^*)) = 0. \tag{32}$$

Unless $V_1(r^*) = V_2(r^*) = t_0$, the seller may have to sell the good to a bidder who would bring a marginal loss to the seller. To see that, suppose $t_0 - V_1(r^*) < 0$. Then Eq. (32) implies $t_0 - V_2(r^*) > 0$. With $t_0 - V_1(r^*) < 0$, the seller's marginal profit from bidder 1 is positive, so she would like to offer more sale to bidder 1. Restricted to a single reserve price for both bidders, however, the seller cannot do that without simultaneously offering more sale to bidder 2, to whom the seller is losing marginally since $t_0 - V_2(r^*) > 0$. So her optimal reserve has to strike a balance between the marginal profit and loss, instead of eliminating the marginal loss.

C The proof of Proposition 2

Let $F := F_i$, as F_i is identical for all i and consider any type $t \leq t^*$. The probability that a bidder of type t obtains the good in the value-bidding equilibrium is $F(t)^{n-1}$. Hence, by the envelope theorem, type t's payoff in the value-bidding equilibrium is

$$U^{\text{val}}(t) \quad = \quad \int_0^t F(x)^{n-1} \mathrm{d}x.$$

The probability that a designated winner of type $x \leq t^*$ gets the good in a t^*-equilibrium is

$$q_w(x) \quad = \quad F(V_*^{-1}(x))^{n-1}, \tag{33}$$

where V_*^{-1} denotes the inverse of the (post-auction) virtual utility function V_* defined by

$$V_*(x) \quad := \quad x - \frac{1 - \frac{F(x)}{F(t^*)}}{\frac{f(x)}{F(t^*)}} = x - \frac{F(t^*) - F(x)}{f(x)} \qquad (x \in [0, t^*]). \tag{34}$$

To understand (33), observe that bidder 1 wins the auction if and only if all other bidders' types are below t^* and, given that she wins, finally keeps the good if and only if all other bidders' virtual utilities are below x, that is, if and only if all other bidders' types are below $V_*^{-1}(x)$.

The payoff of a designated winner of type t^* in a t^*-equilibrium equals $t^* F(t^*)^{n-1}$, because she obtains the good at price 0 and consumes it whenever she wins the auction. Together with (33), the envelope theorem implies that the payoff of a designated winner of type $t \leq t^*$ in a t^*-equilibrium is

$$
\begin{aligned}
U_w(t) &= t^* F(t^*)^{n-1} - \int_t^{t^*} q_w(x)\mathrm{d}x \\
&= t^* F(t^*)^{n-1} - \int_t^{t^*} F(V_*^{-1}(x))^{n-1}\mathrm{d}x. \qquad (35)
\end{aligned}
$$

The probability that a designated loser of type $x \leq t^*$ gets the good in a t^*-equilibrium is

$$
q_l(x) = F(V_*(x))F(x)^{n-2}, \qquad (36)
$$

because she obtains the good if and only if, given the information that her type is below t^*, her virtual utility $V_*(x)$ exceeds the designated winner's type, and all remaining bidders' types are below x. By (36) and the envelope theorem, the payoff for a designated loser of type $t \leq t^*$ in a t^*-equilibrium is

$$
U_l(t) = \int_0^t q_2(x)\mathrm{d}x = \int_0^t F(V_*(x))F(x)^{n-2}\mathrm{d}x. \qquad (37)
$$

Putting (35) and (37) together, the payoff of type $t \leq t^*$ in a t^*-collusive equilibrium is

$$
U^{\mathrm{col}}(t) = \frac{1}{n}U_w(t) + \frac{n-1}{n}U_l(t) \qquad (38)
$$

In the following, we will consider t^*-collusive equilibria with t^* close to 0. To this end, the linearization of F at 0 will be very useful:

$$
F(x) = f(0)x + h_1(x), \qquad (x \geq 0), \qquad (39)
$$

where $|h_1(x)|/x \to 0$ as $x \to 0$. Also, for any $k \geq 0$, we will use $o((t^*)^k)$ to denote any function $h(x,t^*)$ $h(t,t^*))$ such that $\sup_{x\in[0,t^*]} |h(x,t^*)|/(t^*)^k \to 0$ as $t^* \to 0$.

The proof proceeds via a series of lemmas. The next lemma approximates the payoffs in a value-bidding equilibrium.

Lemma 9 *For all t^* and all $t \leq t^*$,*

$$U^{val}(t) \;\; = \;\; f(0)^{n-1}\frac{1}{n}t^n + o((t^*)^n).$$

Proof. From (33) and (39),

$$\begin{aligned} U^{\mathrm{val}}(t) &= \int_0^t F(x)^{n-1}\mathrm{d}x \\ &= \int_0^t (f(0)x + h_1(x))^{n-1}\mathrm{d}x \\ &= \int_0^t \left(f(0)^{n-1}(x)^{n-1} + h_2(x)\right)\,\mathrm{d}x \\ &= f(0)^{n-1}\frac{1}{n}t^n + \int_0^t h_2(x)\,\mathrm{d}x, \end{aligned}$$

where $|h_2(x)|/(x)^{n-1} \to 0$ as $x \to 0$.

Let $\epsilon > 0$. If t^* is sufficiently small, then $|h_2(x)| \leq \epsilon\ (x)^{n-1}$ for all $x \leq t^*$. Therefore

$$\left| \int_0^t h_2(x)\mathrm{d}x \right| \leq \int_0^t \ |h_2(x)|\,\mathrm{d}x \leq \epsilon \int_0^t x^{n-1}\mathrm{d}x \leq \epsilon\ (t^*)^n,$$

which completes the proof. ■

Define

$$\kappa(t^*) := \frac{f(0)}{\min_{x\in[0,t^*]} f(x)} \qquad (t^* \in [0,\bar{t}]). \tag{40}$$

Observe that $\kappa(t^*) \to 1$ as $t^* \to 0$.

The next lemma provides an approximate lower bound for the virtual valuation function if t^* is small.

Lemma 10 *For all $t^* \in [0,\bar{t}]$ and all $t \in [0,t^*]$,*

$$V_*(t) \;\; \geq \;\; t - \kappa(t^*)(t^* - t) + o(t^*).$$

Proof. Using (34) and (39),

$$\begin{aligned} V_*(t) &= t - \frac{f(0)t^* + h_1(t^*) - f(0)t - h_1(t)}{f(t)} \\ &= t - \frac{f(0)}{f(t)}(t^* - t) + h(t,t^*) \\ &\geq t - \kappa(t^*)(t^* - t) + h(t,t^*), \end{aligned} \tag{41}$$

where

$$h(t,t^*) := \frac{h_1(t^*) - h_1(t)}{f(t)}. \tag{42}$$

Observe that (39) implies

$$\frac{\sup_{x\in[0,t^*]} \mid h_1(x) \mid}{t^*} \leq \sup_{x\in[0,t^*]} \frac{\mid h_1(x) \mid}{x} \to 0 \quad \text{as} \quad t^* \to 0.$$

Hence, defining $\underline{f}(t^*) := \min_{x\in[0,t^*]} f(x)$, we have

$$\frac{\sup_{x\in[0,t^*]} |h(x,t^*)|}{t^*} \leq \frac{1}{\underline{f}(t^*)}\left(\frac{|h_1(t^*)|}{t^*} + \frac{\sup_{x\in[0,t^*]} |h_1(x)|}{t^*}\right) \to 0$$

as $t^* \to 0$. ■

Next we establish a result concerning the inverse virtual valuation function that is analogous to Lemma 10.

Lemma 11 *For all $t^* \in [0,\bar{t}]$ and all $t \in [0,t^*]$,*

$$V_*^{-1}(t) \leq \frac{t + \kappa(t^*)t^*}{1 + \kappa(t^*)} + o(t^*).$$

Proof. Let h be defined as in (42). Rewriting (41) in inverse form,

$$\begin{aligned} t &\geq V_*^{-1}(t) - \kappa(t^*)(t^* - V_*^{-1}(t)) + h(V_*^{-1}(t),t^*) \\ &= V_*^{-1}(t)(1 + \kappa(t^*)) - \kappa(t^*)t^* - y(t,t^*)(1 + \kappa(t^*)), \end{aligned} \tag{43}$$

where

$$y(t,t^*) \quad := \quad -\frac{h(V_*^{-1}(t),t^*)}{1+\kappa(t^*)}. \tag{44}$$

Solving (43) for $V_*^{-1}(t)$ yields

$$V_*^{-1}(t) \quad \leq \quad \frac{t+\kappa(t^*)t^*}{1+\kappa(t^*)} + y(t,t^*).$$

Using (44) and the fact that $V_*^{-1}(t) \in [0,t^*]$,

$$\frac{\sup_{x\in[0,t^*]} \ |y(x,t^*)|}{t^*} \leq \frac{\sup_{x\in[0,t^*]} \ |h(x,t^*)|}{(1+\kappa(t^*))t^*} \to 0 \quad \text{as} \quad t^* \to 0,$$

because $h(x,t^*) = o(t^*)$. ■

For all $y \in [0,1]$ and $k > 0$, let

$$\phi_1(y,k) \quad = \quad 1 - \frac{1+k}{n} + \frac{(y+k)^n}{n(1+k)^{n-1}}.$$

Lemma 12 *For all $t^* \in [0,\bar{t}]$ and all $t \in [0,t^*]$,*

$$U_w(t) \quad \geq \quad f(0)^{n-1}(t^*)^n\phi_1(\frac{t}{t^*},\kappa(t^*)) + \ o((t^*)^n).$$

Proof. Using (39) and Lemma 11,

$$\begin{aligned}
&\int_t^{t^*} F(V_*^{-1}(x))^{n-1}\mathrm{d}x \\
&= \int_t^{t^*} \left(f(0)V_*^{-1}(x) + h_1(V_*^{-1}(x))\right)^{n-1} \mathrm{d}x \\
&\leq \int_t^{t^*} \left(f(0)\frac{x+\kappa(t^*)t^*}{1+\kappa(t^*)} + o(t^*)\right)^{n-1} \mathrm{d}x \\
&= f(0)^{n-1} \int_t^{t^*} \left(\frac{x+\kappa(t^*)t^*}{1+\kappa(t^*)}\right)^{n-1} \mathrm{d}x \ + o((t^*)^n) \\
&= f(0)^{n-1} \left(\frac{1+\kappa(t^*)}{n}(t^*)^n - \frac{(t+\kappa(t^*)t^*)^n}{n(1+\kappa(t^*))^{n-1}}\right) \ + o((t^*)^n) \\
&= f(0)^{n-1}(t^*)^n \left(\frac{1+\kappa(t^*)}{n} - \frac{(\frac{t}{t^*}+\kappa(t^*))^n}{n(1+\kappa(t^*))^{n-1}}\right) \ + o((t^*)^n).
\end{aligned}$$

Hence, the result follows from (35) and the fact that $t^*F(t^*)^{n-1} = f(0)^{n-1}(t^*)^n + o((t^*)^n)$. ■

For all $y \in [0,1]$ and $k > 0$, let

$$\phi_2(y,k) \ := \ \frac{1+k}{n}y^n - \frac{k}{n-1}y^{n-1} + \frac{k^n}{n(n-1)(1+k)^{n-1}} \ \text{ if } y > \frac{k}{1+k}, \tag{45}$$

and otherwise $\phi_2(y,k) := 0$. It is easy to check that ϕ_2 is continuous.

Lemma 13 *For all $t^* \in [0,\bar{t}]$ and all $t \in [0,t^*]$,*

$$U_l(t) \ \geq \ f(0)^{n-1}(t^*)^n\phi_2(\frac{t}{t^*},\kappa(t^*)) + \ o((t^*)^n).$$

Proof. Using (39),

$$F(V_*(x)) \ = \ f(0)V_*(x) + h_1(V_*(x)) \quad \text{ if } V_*(x) > 0,$$

and $F(V_*(x)) = 0$ otherwise. Hence, using that $V_*(x) \in [0,t^*]$ if $V_*(x) > 0$,

$$F(V_*(x)) \ = \ f(0)\max\{0, V_*(x)\} + o(t^*).$$

Therefore, using Lemma 10,

$$F(V_*(x)) \;\geq\; f(0)\max\{0, x-\kappa(t^*)(t^*-x)\} + o(t^*).$$

Hence,

$$\begin{aligned}&\int_0^t F((V_*)(x))F(x)^{n-2}\mathrm{d}x\\ &\geq f(0)^{n-1}\int_0^t \max\{0, x-\kappa(t^*)(t^*-x)\}x^{n-2}\mathrm{d}x \;+o((t^*)^n).\end{aligned} \tag{46}$$

Observe that the following equivalence holds:

$$x-\kappa(t^*)(t^*-x) > 0 \quad\Leftrightarrow\quad x > \frac{\kappa(t^*)}{1+\kappa(t^*)}t^*.$$

Suppose first that $t/t^* > \kappa(t^*)/(1+\kappa(t^*))$. Using (37) and (46),

$$\begin{aligned}U_l(t) &\geq f(0)^{n-1}\int_{\frac{\kappa(t^*)}{1+\kappa(t^*)}t^*}^t (x-\kappa(t^*)(t^*-x))(x)^{n-2}\mathrm{d}x \;+o((t^*)^n)\\ &= f(0)^{n-1}(t^*)^n\left(\frac{1+\kappa(t^*)}{n}(\frac{t}{t^*})^n + \frac{\kappa(t^*)^n}{n(n-1)(1+\kappa(t^*))^{n-1}}\right.\\ &\qquad\left.-\frac{\kappa(t^*)}{n-1}(\frac{t}{t^*})^{n-1}\right) \qquad +o((t^*)^n)\\ &= f(0)^{n-1}(t^*)^n\phi_2(\frac{t}{t^*},\kappa(t^*)) + o((t^*)^n).\end{aligned}$$

If $t/t^* < \kappa(t^*)/(1+\kappa(t^*))$, then (37) and (46) yield

$$\begin{aligned}U_l(t) &\geq o((t^*)^n)\\ &= f(0)^{n-1}(t^*)^n\phi_2(\frac{t}{t^*},\kappa(t^*)) + \; o((t^*)^n),\end{aligned}$$

because $\phi_2(t/t^*,\kappa(t^*)) = 0$. ■

Lemma 14 *If k is sufficiently close to 1, then*

$$\min_{y\in[0,1]} \frac{1}{n}\phi_1(y,k) + \frac{n-1}{n}\phi_2(y,k) - \frac{1}{n}y^n > 0.$$

Proof. By continuity, it is sufficient to consider $k = 1$; i.e., to show for all $y \in [0,1]$,

$$\frac{1}{n}\phi_1(y,1) + \frac{n-1}{n}\phi_2(y,1) - \frac{1}{n}y^n > 0. \tag{47}$$

Defining

$$\psi(y) := \phi_1(y,1) - y^n = 1 - \frac{2}{n} + \frac{(y+1)^n}{n\ 2^{n-1}} - y^n,$$

we have

$$\psi'(y) = \frac{(y+1)^{n-1}}{2^{n-1}} - ny^{n-1} = y^{n-1}\underbrace{\left(\frac{1}{2^{n-1}}(1+\frac{1}{y})^{n-1} - n\right)}_{\text{strictly decreasing in } y}.$$

Hence, there exists y^* such that $\psi'(y) > 0$ if $y < y^*$ and $\psi'(y) < 0$ if $y > y^*$. Therefore, ψ takes its minimum on $[0,1]$ at 0 or at 1. Because $\psi(0) > 1 - 2/n \geq 0$ and $\psi(1) = 0$,

$$\frac{1}{n}\phi_1(y,1) - \frac{1}{n}y^n = \frac{1}{n}\psi(y) > 0 \quad \text{if } \ y < 1. \tag{48}$$

From (45),

$$\frac{\partial}{\partial y}\phi_2(y,1) = 2y^{n-1} - y^{n-2} = y^{n-2}(2y-1) > 0 \quad \text{if } \ y > \frac{1}{2}.$$

Hence, if $y > 1/2$, then $\phi_2(y,1) > 0$, because $\phi_2(\frac{1}{2},1) = 0$. Combining this with (48) and the fact that $\phi_2(y,1) = 0$ if $y < 1/2$, we obtain (47). ■

For all $t^* \in [0,\bar{t}]$ and all $t \in [0,t^*]$, using (38), Lemmas 9, 12, and 13,

$$\frac{U^{\text{col}}(t) - U^{\text{val}}(t)}{f(0)^{n-1}(t^*)^n} \geq \frac{1}{n}\phi_1(\frac{t}{t^*},\kappa(t^*)) + \frac{n-1}{n}\phi_2(\frac{t}{t^*},\kappa(t^*)) - \frac{1}{n}(\frac{t}{t^*})^n + o(1). \tag{49}$$

If t^* is sufficiently close to 0, then $\kappa(t^*)$ is arbitrarily close to 1, and thus (49) together with

Lemma 14 implies that

$$\min_{t\in[0,t^*]}(U^{\text{col}}(t) - U^{\text{val}}(t)) > 0.$$

This together with Lemma 6 proves Proposition 2.

D References

Blume, A. and P. Heidhues (2004): "All Equilibria of the Vickey Auction," *Journal of Economic Theory*, 114, 170-177.

Brusco, S. and G. Lopomo (2002): "Collusion via Signalling in Simultaneous Ascending Bid Auctions with Heterogeneous Objects, with and without Complementarities," *Review of Economic Studies*, 69, 407–436.

— (2006): "Simultaneous Ascending Bid Auctions with Privately Known Budget Constraints," *Journal of Industrial Economics*, forthcoming.

Calzolari, G., and A. Pavan (2006): "Monopoly with Resale," *RAND Journal of Economics*, 37(2), 362–375.

Cass, D., and K. Shell (1983): "Do Sunspots Matter?" *Journal of Political Economy*, 91(2), 193–227.

Che, Y.-K. and J. Kim (2006), "Robustly Collusion-Proof Implementation," *Econometrica*, 74(4), 1063–1107.

Cramton, P., and R. Gibbons and P. Klemperer (1987): "Dissolving a Partnership Efficiently," *Econometrica*, 55(3), 615–632.

Engelbrecht-Wiggans, R. and C. M. Kahn (2005): "Low-Revenue Equilibria in Simultaneous Ascending-Bid Auctions," *Management Science*, 51, 508–518.

Garratt, R. and T. Tröger (2006): "Speculation in Standard Auctions with Resale," *Econometrica*, 74, 753–769.

Graham, D. A. and R. C. Marshall (1987): "Collusive Bidder Behavior at Single-Object Second-Price and English Auctions," *Journal of Political Economy*, 95, 1217–1239.

Hafalir, I. and V. Krishna (2007): "Asymmetric Auctions with Resale" *American Economic Review*, forthcoming.

Haile, P. (1999): "Auctions With Resale," mimeo, University of Wisconsin, Madison.

— (2003): "Auctions with Private Uncertainty And Resale Opportunities," *Journal of Economic Theory*, 108, 72–100.

Laffont, J.-J., and D. Martimort (2000): "Mechanism Design with Collusion and Correlation," *Econometrica*, 68(2), 309–342.

Lebrun, B. (2007): "First-Price and Second-Price Auctions with Resale," mimeo.

Mailath, G. J. and P. Zemsky (1991): "Collusion in Second Price Auctions with Heterogeneous Bidders," *Games and Economic Behavior*, 3, 467–486.

McAfee, R. P. and J. McMillan (1992): "Bidding Rings," *American Economic Review*, 82, 579–599.

Milgrom, P. R. (2000): "Putting Auction Theory to Work: The Simultaneous Ascending Auction," *Journal of Political Economy*, 108, 245–272.

Milgrom, P. R. and R. J. Weber (1982): "A Theory of Auctions and Competitive Bidding," *Econometrica*, 50, 1089–1122.

Myerson, R. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58–73.

Mylovanov, T. and T. Tröger (2006): "A Characterization of the Conditions for Optimal Auction with Resale," mimeo.

Pagnozzi, M. (2007): "Should Speculators Be Welcomed in Auctions?" Centre for Studies in Economics and Finance (CSEF), University of Salerno, CSEF Working Paper 176.

Robinson, M. S. (1985): "Collusion and the Choice of Auction," *RAND Journal of Economics*, 16, 141-145.

Shell, K. (1977): "Monnaie et Allocation Intertemporelle," mimeo, Seminaire Roy-Malinvaud, Centre National de la Recherche Scientifique, Paris, November 21.

Smith, Richard A. (1961): "The Incredible Electrical Conspiracy (Part II)," *Fortune*, May.

Zheng, C. (2000): "An Optimal Auction When Resale Cannot Be Prohibited," Northwestern University, CMS-EMS Discussion Paper 1303.

— (2002): "Optimal Auction with Resale," *Econometrica*, 70, 2197–2224.