Sharing a Risky Cake*

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Abstract

Consider an n-person Nash bargaining problem where players bargain over the division of a cake whose size is stochastic. In such a game, the players are not only bargaining over the division of a cake, but they are also sharing risk. This paper presents the Nash bargaining solution to this problem, investigates its properties, and highlights a few special cases.

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1 Introduction

Imagine a game where rational players bargain over a set of outcomes. The payoffs are determined by mutual agreement between the players. In a seminal paper, Nash (1950) devised a simple axiomatic characterization of the solution to this general bargaining problem. Nash's paper has since spawned a vast literature on bargaining theory and its applications.

In this paper, we investigate the case where the set of outcomes is the utility possibility frontier arising from the division of a cake whose size is stochastically determined. Players must agree on how to divide the cake before the cake's size is known. Intuitively, in such a situation, we would expect risk-averse players to forgo larger shares of the cake when the cake is big to insure themselves against the possibility that the cake is small. On the other hand, we would expect risk-loving players to gamble away small stakes for a chance at having large amounts of cake. This game can serve as an allegory for many "real" bargaining situations, particularly those where an unknown amount of money, brought about through cooperation, is to be divided.

We study the Nash bargaining solution in such a game and find that it confirms this intuition. We characterise and investigate the properties of the solution for the case where n-players with different beliefs bargain over a one-dimensional random variable. The contribution of this paper is solving and analysing the properties of the Nash bargaining solution without presupposing that the players hold the same beliefs or that the cake is a discrete random variable.

2 The Nash Bargaining Problem

Definition 2.1. For each natural number n, an **n-person Nash bargaining problem** is any list (S, d), where S is a nonempty, convex, and compact subset of $\mathbb{R}^n$ and d is an element of S such that there exists at least one $x \in S$ with $x \gg d$. Here, $\gg$ denotes the natural partial order on $\mathbb{R}^n$ — that is, $x \gg y$ if and only if each component of x is greater than or equal to y.

The set S represents the utility possibility set that arises from agreement between the players and d denotes the outcome if the players fail to reach an agreement.

Definition 2.2. Let (S, d) be a Nash bargaining problem. The **Nash bargaining solution** is a point $c(S, d)$ in S that satisfies the following conditions:

(PE) The solution $c(S, d)$ is **Pareto-efficient**; that is, if some outcome x is Pareto-superior to $c(S, d)$, then $x \notin S$.

(AM) The solution $c(S, d)$ is invariant to equivalent representations of utility functions; that is, transforming S by an increasing affine transformation results in an equivalent transformation of the Nash bargaining solution.

(IIA) The solution is independent of irrelevant alternatives; that is, if $T \subseteq S$ and $c(S, d) \in T$, then $c(T, d) = c(S, d)$.

(SYM) If S is symmetric and all players face the same disagreement outcome, then $c(S, d)$ assigns the same utility to all players.

There is an extensive literature on the appropriateness of these axioms (see Muthoo, 1999). We simply take them as given in this paper.

The following theorem, originally proved for the 2-person case by Nash (1950), and generalised to the n-person case by Harsanyi (1997), forms the basis for our results.

Theorem 2.1 (Nash)**.** *For each n-person Nash bargaining problem (S, d), the Nash bargaining solution $c(S, d)$ exists, is unique, and*

$$c(S,d) \in \underset{\boldsymbol{V}\in S}{\operatorname{argmax}} \prod_{i=1}^{n}(V_i - d_i), \tag{1}$$

*where V_i is the ith component of the vector $\boldsymbol{V}$ in S. The product in (1) is called the **Nash product**.*

Suppose that a deterministic one-dimensional cake of size x is shared between players 1 and 2. Denote player i's utility function by V_i and let

$$\tilde{w} \in \underset{0\le w\le x}{\operatorname{argmax}}(V_1(w) - d_1)(V_2(x-w) - d_2).$$

Then theorem 2.1 implies that $(V_1(\tilde{w}), V_2(x - \tilde{w}))$ is the Nash bargaining solution. In this case, the cake is partitioned into $\tilde{w}$ and $x - \tilde{w}$. In the next section, we investigate the stochastic analogue to this problem.

3 The Nash Bargaining Problem with Risk

For clarity of exposition and simplicity of notation, we begin with a restricted version of the problem and make generalisations as we progress. First, we study the case where 2 players with identical beliefs bargain over a random variable. Then we extend the analysis to the case where the two players have differing beliefs. Finally, we turn our attention to the n-player case with differing beliefs. We supplement the results throughout with convenient numerical examples. All proofs, along with the necessary mathematical machinery, are relegated to the appendix.

To begin with, suppose that players 1 and 2 bargain over the partition of a random cake X (a random variable) with an associated probability distribution F (this may correspond to a continuous, discrete or mixed distribution function). Further suppose that F has support $\Omega \subset \mathbb{R}$. The contract $\tilde{w}$ corresponding to the Nash bargaining solution must then be the solution to the following optimisation problem:

$$\max_{w}(\mathrm{E}[u_1(w(x))] - d_1)(\mathrm{E}[u_2(x - w(x))] - d_2), \tag{2}$$

where $\mathrm{E}[\,\cdot\,]$ denotes the expectation operator, x the total amount of cake and $w(x)$ the amount of cake allocated to player 1 given $X = x$. Due to axiom (AM), we can assume without loss of generality that the disagreement point (d_1, d_2) is equal to the origin. For this problem, we can prove the following theorem.

Theorem 3.1. *For a 2-person Nash bargaining problem where a random variable X is being shared, the contract $\tilde{w}$, denoting the amount of cake given to player 1 given the outcome, must satisfy*

$$\frac{u_1'(\tilde{w}(x))}{u_2'(x-\tilde{w}(x))} = \frac{E[u_1(\tilde{w}(y))]}{E[u_2(y-\tilde{w}(y))]} \quad (F-\textit{almost everywhere}). \tag{3}$$

Here u_i' denotes the derivative of u with respect to its argument. The phrase "F-almost everywhere" means that (3) need only hold on sets of nonzero probability. In other words, if we define the event B by

$$B := \left\{ x \in \Omega : \frac{u_1'(\tilde{w}(x))}{u_2'(x-\tilde{w}(x))} \neq \frac{\mathrm{E}[u_1(\tilde{w}(y))]}{\mathrm{E}[u_2(y-\tilde{w}(y))]} \right\},$$

then, $F(B) = 0$. Observe that the right hand side of (3) is a *constant* in x — that is, the contract $\tilde{w}$ which solves the Nash bargaining problem must equate the ratio of the marginal utilities to a constant number for every outcome. This is a first order condition for the optimisation problem.

This theorem is our work-horse, it allows us to explicitly solve for the solution of many numerical examples. It also enables us to investigate the properties of the solution in the general case. Unfortunately, the notation is a little awkward and hard to interpret, so it is helpful to consider a specific example to sharpen our intuition about what the result actually implies. Before we do this, note that (3) implies that the distribution of the cake does not affect the functional form of the solution.[1]

Example 3.1. Consider the case where the utility functions u_1 and u_2 of player 1 and 2 are

$$u_1(x) := \sqrt{x} \text{ and } u_2(x) := \log(1+x).$$

Suppose further that the cake is continuously distributed and its support is some set Ω of non-negative real numbers. By theorem 3.1, the contract path $\tilde{w}$, specifying player 1's share of cake, will satisfy

$$\frac{\tilde{w}(x)-x-1}{2\sqrt{\tilde{w}(x)}} = \frac{\mathrm{E}(\sqrt{\tilde{w}(y)})}{\mathrm{E}(\log y - \tilde{w}(y))} \quad \text{(almost everywhere)}.$$

Now note that the right hand side is a constant in x. Let

$$\alpha := \frac{\mathrm{E}(\sqrt{\tilde{w}(y)})}{\mathrm{E}(\log(y-\tilde{w}(y)))},$$

and solve for $\tilde{w}$ to get

$$\tilde{w}(x) = 2\alpha^2 + 1 + x + 2\alpha\sqrt{\alpha^2+1+x}.$$

This pins down the functional form of the contract curve. If we now specify a distribution for the cake, we can numerically solve for the value of the constant α. □

Proposition 3.2. *Suppose that one party is risk-neutral, the other is risk-averse, and that the hypotheses of theorem 3.1 are satisfied. Then the amount of cake allocated to the risk-averse individual will be some fixed amount regardless of the outcome.*

[1]As we shall see later, this observation fails to hold when the players have differing beliefs.

This is effectively an insurance contract between the risk-averse and the risk-neutral party. This result should not come as a surprise and its appearance is a direct consequence of the Pareto Efficiency axiom (PE). The work of Borch (1962) on risk sharing informs us that all Pareto-efficient risk-sharing allocations, such as this bargaining one, will have this property.

We can do more to relate the Nash bargaining solution to the work of Borch. In the two-person case, Borch's celebrated result states that to each efficient allocation of risk there corresponds two positive real numbers λ_1 and λ_2 such that the efficient allocation maximises the function defined by:

$$\lambda_1 \mathrm{E}(u_1(w(x))) + \lambda_2 \mathrm{E}(u_2(x - w(x))).$$

The two scalars λ_1 and λ_2 trace out the Pareto frontier for the problem; it is routine to prove the following.

Proposition 3.3. *For the Nash bargaining solution, the Borch constants λ_1 and λ_2 are equal to $E(u_1(\tilde{w}(y)))$ and $E(u_1(y - \tilde{w}(y)))$ respectively, where $\tilde{w}$ is a contract that maximises the Nash product.*

The next result focuses on the bargaining aspect of the problem.

Proposition 3.4. *If the risk aversion of the risk-averse party in proposition 3.2 increases, then the expected amount of cake he receives decreases.*

The following conjecture is a related but more general result.

Conjecture 3.1. *If the risk aversion of a player increases, then their expected share of the cake decreases.*

Here, we do not assume that the second party is risk-neutral. The result seems plausible but a proof remains elusive. Note that by "expected share of the cake", we refer to the expectation of the contract rather than the contract itself. In particular, if the risk aversion of a party increases, this does *not* necessarily imply that their share of the cake becomes smaller everywhere.

3.1 Differing Beliefs

In order to reason about players with different beliefs, we need to rule out a potential pathology. Let F_i be the probability distribution associated with player i's beliefs about the cake and let $\mathfrak{B}$ denote the event space. We assume that

$$\forall_{(B \in \mathfrak{B})} \left(F_i(B) = 0 \Longleftrightarrow F_j(B) = 0\right). \tag{4}$$

In words, if one player believes that an event will occur with probability zero, then so too does the other. This assumption rules out the degenerate case where one player believes an event will occur with positive probability while the other believes it will occur with zero probability. Intuitively, if this were to happen, then axiom (PE) would dictate that all the cake be given to the first player in that event. We choose to ignore uninteresting degenerate cases like this and focus instead on the case where players assign nonzero probabilities to the same events.

Note that (4) is not critical to our analysis; we can discard it simply by invoking the Lebesgue decomposition theorem. This theorem allows us to decompose F_i into F_i^1 and F_i^2, where F_i^1 is absolutely continuous with respect to F_j and F_i^2 is singular with respect to F_j. We can then carry out our analysis on each part separately. This exercise would complicate the analysis but yield no new insights.

Before we begin, we introduce a useful concept from measure theory. The *Radon-Nikodym derivative* of F_2 with respect to F_1 is a measurable function g that satisfies

$$F_2(A) = \int_A g \, \mathrm{d}F_1 \tag{5}$$

for every event A. Some authors write g as $\mathrm{d}F_2/\mathrm{d}F_1$, though one should be aware that, in general, this is not a conventional derivative. The existence of the Radon-Nikodym derivative is guaranteed by the Radon-Nikodym theorem. In the case where F_1 and F_2 are distributions of continuous random variables, the Radon-Nikodym derivative of F_2 with respect to F_1 is just the ratio of their density functions. This can be verified by (5):

$$F_2(A) = \int_A f_2(t) \, \mathrm{d}t = \int_A \frac{f_2(t)}{f_1(t)} f_1(t) \, \mathrm{d}t = \int_A \frac{f_2(t)}{f_1(t)} \, \mathrm{d}F_1.$$

Example 3.2. Let $\Omega = \{1, 2, 3\}$ and consider the following probability mass functions

$$p_1(x) = \begin{cases} 1/3 & : x = 1 \\ 1/3 & : x = 2 \\ 1/3 & : x = 3 \end{cases}, \quad p_2(x) = \begin{cases} 1/2 & : x = 1 \\ 1/4 & : x = 2 \\ 1/4 & : x = 3 \end{cases}.$$

Let P_1 and P_2 denote the distributions associated with these mass functions. Then the Radon-Nikodym derivative g of P_1 with respect to P_2 is given by

$$g(x) = \begin{cases} 2/3 & : x = 1 \\ 4/3 & : x = 2 \\ 4/3 & : x = 3 \end{cases}.$$

Theorem 3.5. *Consider a 2-person Nash bargaining problem where a random variable X is being shared. Suppose that the probability measures F_1 and F_2 denote the beliefs of the players, and that (4) holds. Then the contract $\tilde{w}$, denoting the amount of cake allocated to player 1, must satisfy*

$$\frac{u_1'(\tilde{w}(x))}{u_2'(x - \tilde{w}(x))} = \frac{E[u_1(\tilde{w}(y))]}{E[u_2(y - \tilde{w}(y))]g(x)} \quad (F_1 - \textit{almost everywhere}), \tag{6}$$

where g is the Radon-Nikodym derivative of F_2 with respect to F_1.

In the case where both players believe that the cake is a continuous random variable with density function f_i, condition (6) can be written as

$$\frac{u_1'(\tilde{w}(x))}{u_2'(x - \tilde{w}(x))} = \frac{E[u_1(\tilde{w}(y))]f_2(x)}{E[u_2(y - \tilde{w}(y))]f_1(x)} \quad (\textit{almost everywhere}).$$

The introduction of differing beliefs adds a great deal of realism to the problem. We see that beliefs can offset risk aversion — that is, an optimistic risk-averse player can take on more risk than a pessimistic risk-loving player. No longer is there necessarily an "insurance" contract between risk-averse and risk-neutral players. We can still however say the following:

Proposition 3.6. *Given the hypothesis of theorem 3.5, suppose that one party is risk-averse and the other is risk-neutral. If the risk aversion of the risk-averse party increases, then the expected amount of cake he receives decreases.*

This proposition directly generalises proposition 3.4 to the case where the players have different beliefs.

3.2 n-Players

Finally, we generalise our result to the n-person case. The result, though a direct extension, is much more cumbersome to write.

Theorem 3.7. *Consider an n-person Nash bargaining problem where a random variable X is being shared. Suppose that the probability measure F_i denotes player i's beliefs, and that (4) holds. Then the contract vector $(\tilde{w}_1(x), \ldots, \tilde{w}_n(x))$ denoting the amount of cake allocated to players 1 through n for cake size x will satisfy*

$$\frac{u_i'(\tilde{w}_1(x))}{u_n'(\tilde{w}_n(x))} = \frac{E[u_i(\tilde{w}_i(y))]}{E[u_n(\tilde{w}_n(y))]g_i(x)} \quad (F_i - \textit{almost everywhere}) \tag{7}$$

for every $i = 1, \ldots, n-1$, where g_i is the Radon-Nikodym derivative of F_n with respect to F_i. We also have that $\tilde{w}_n(x) \equiv x - \sum_{i=1}^{n-1} \tilde{w}_i(x)$.

In the case where all players believe the cake is a continuous random variable, we can rewrite (7) as

$$\frac{u_i'(\tilde{w}_i(x))}{u_n'(\tilde{w}_n(x))} = \frac{E[u_i(\tilde{w}_i(y))]f_n(x)}{E[u_n(\tilde{w}_n(y))]f_i(x)} \quad (\textit{almost everywhere}),$$

where f_i is the density function characterizing player i's beliefs.

As expected, (7) gives $n-1$ first-order conditions.

4 Appendix: Proofs

Before we begin, we need some ancillary results from nonlinear analysis. For a more detailed treatment of this, consult a specialised text like Zeidler (1995, Chap 4.). If you are familiar with even the basic properties of Gâteaux derivatives, the following subsection can be skipped.

4.1 Mathematical Machinery

Definition 4.1. Let $(X, \|\cdot\|_{\mathrm{X}})$ and $(Y, \|\cdot\|_{\mathrm{Y}})$ be two normed linear spaces, and T a subset of X. For any x_0 in the interior of T, a map $f : T \to Y$ is said to be **Gâteaux differentiable** at x_0 if there is a bounded linear operator δ_{f,x_0} such that

$$\lim_{t \to 0} \frac{\|f(x_0 + th) - f(x_0) - \delta_{f,x_0}(th)\|_{\mathrm{Y}}}{t} = 0 \tag{8}$$

for all $h \in X$. We call the linear operator δ_{f,x_0} the **Gâteaux derivative** of f at x_0.

Proposition 4.1. *Let S be an open subset of a normed linear space X, and f be a mapping from S into $\mathbb{R}$. Suppose that f is Gâteaux differentiable at the point $x \in S$. Then the functional*

$$\frac{d}{dt} f(x+th)\bigg|_{t=0}$$

is the Gâteaux derivative of f at x if and only if it is a linear functional of h.

Theorem 4.2 (Fermat's Theorem). *A necessary condition for a Gâteaux differentiable functional Φ to have an extremum at $\hat{x}$ is that the Gâteaux derivative $\delta_{\Phi,\hat{x}}$ be the zero operator.*

4.2 Proof of Claims

With the aid of the above results, in particular Fermat's theorem (theorem 4.2), we can set to work proving the claims made in this paper.

Proof of theorem 3.7. Let $w = (w_1, \ldots, w_n)$ be each player's share of the cake. Since the shares must add to the total amount of cake available, we can write

$$w_n(x) \equiv x - \sum_{i=1}^{n-1} w_i(x).$$

By translating utility functions and using theorem 2.1, we simply need to maximise the Nash product

$$J(w) := \prod_{i=1}^{n} \mathrm{E}(u_i(w_i(x))),$$

to prove the theorem.

Let $h = (h_1, \ldots, h_{n-1}, 0)$, where each h_i is a real-valued function of x. By proposition 4.1 and the product rule, the Gâteaux derivative of J is given by:

$$\begin{aligned}
\delta_{J,w}(h) &= \frac{\mathrm{d}}{\mathrm{d}t} J(w+th)\bigg|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{E}(u_n(w_n - t\sum^{n-1} h_i)) \prod_{i=1}^{n-1} \mathrm{E}(u_i(w_i + th_i))\right)\Bigg|_{t=0} \\
&= \sum_{i=1}^{n-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_i(w_i + th_i)\,\mathrm{d}F_i(x)\left(\prod_{j\neq i}^{n-1} \mathrm{E}(u_j(w_j + th_j))\right)\mathrm{E}(u_n(w_n(x) - t\sum_{i=1}^{n-1} h_i))\right) \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u_n(w_n - t\sum_{i=1}^{n-1} h_i)\,\mathrm{d}F_n(x)\right)\prod_{j\neq i}^{n-1} \mathrm{E}(u_j(w_j))\Bigg|_{t=0},
\end{aligned}$$

By invoking Leibniz's rule for Lebesgue integrals and evaluating at $t = 0$, we can simplify the above to

$$\begin{aligned}\delta_{J,w}(h) = \sum_{i=1}^{n-1} &\left(\mathrm{E}(u_n(w_n)) \prod_{j \neq i}^{n-1} \mathrm{E}(u_j(w_j)) \int_\Omega \left[u_i'(w_i(x)) h_i(x)\, \mathrm{d}F_i(x) \right] \right) \\ &- \prod_{j=1}^{n-1} \mathrm{E}(u_j(w_j)) \int_\Omega \left[u_n'(w_n(x)) \sum_{i=1}^{n-1} h_i(x) \right] \mathrm{d}F_n(x),\end{aligned}$$

where u_i' denotes the derivative of u_i. By theorem 4.2, the Gâteaux derivative of J at the optimum contract $\tilde{w}$ is identically zero. In particular, $\delta_{J,\tilde{w}} = 0$ when $h_k = 0$ for every $j \neq i$. Using this fact, we get $n - 1$ first order conditions:

$$\prod_{j \neq i}^{n} \mathrm{E}(u_j(\tilde{w}_j)) \int_\Omega u_i'(\tilde{w}_i(x)) h_i(x)\, \mathrm{d}F_i(x) - \prod_{j=1}^{n-1} \mathrm{E}(u_j(\tilde{w}_j)) \int_\Omega u_n'(x - \sum_{i=1}^{n-1} \tilde{w}_i(x)) h_i(x)\, \mathrm{d}F_n(x) = 0$$

for each $i \in \{1, \ldots, n-1\}$ and every admissible h_i. Since for nontrivial cakes $E(u_k(\tilde{w})) \neq 0\,(1 \leq k \leq n)$, we can divide through on both sides of the equation and write

$$\mathrm{E}(u_n(\tilde{w}_n)) \int_\Omega u_i'(\tilde{w}_i(x)) h_i(x)\, \mathrm{d}F_i(x) - \mathrm{E}(u_i(\tilde{w}_i)) \int_\Omega u_n'(x - \sum_{i=1}^{n-1} \tilde{w}_i(x)) h_i(x)\, \mathrm{d}F_n(x) = 0, \quad (9)$$

for every admissible h_i. By (4) and the Radon-Nikodym theorem, there exists a measurable function g_i, known as the Radon-Nikodym derivative of F_n with respect to F_i, that allows us to write (9) as

$$\begin{aligned} 0 &= \mathrm{E}(u_n(\tilde{w}_n)) \int_\Omega u_i'(\tilde{w}_i(x)) h_i(x)\, \mathrm{d}F_i(x) - \mathrm{E}(u_i(\tilde{w}_i)) \int_\Omega u_n'(x - \sum_{i=1}^{n-1} \tilde{w}_i(x)) g_i(x) h_i(x)\, \mathrm{d}F_i(x) \\ &= \int_\Omega \left[\mathrm{E}(u_n(\tilde{w}_n)) u_i'(w(x)) - \mathrm{E}(u_i(\tilde{w}_i)) u_n'(\tilde{w}_n(x)) g_i(x) \right] h_i(x)\, \mathrm{d}F_i(x). \end{aligned}$$

By the fundamental lemma of the calculus of variations, we infer that

$$E(u_n(\tilde{w}_n)) u_i'(\tilde{w}_i(x)) - E(u_n(\tilde{w}_i)) u_n'(\tilde{w}_n(x)) g_i(x) = 0 \quad (F_i - \text{ almost everywhere}).$$

Recall that the nonnegativity condition implies that $\tilde{w}_n(x) \equiv x - \sum^{n-1} \tilde{w}_i(x)$. So, by rearranging, we conclude that

$$\frac{\mathrm{E}(u_i(\tilde{w}_i(x)))}{\mathrm{E}(u_n(x - \sum^{n-1} \tilde{w}_j(x)))} = \frac{u_i'(w_i(x))}{u_n'(x - \sum^{n-1} \tilde{w}_j(x)) g_i(x)} \quad (F_i - \text{almost everywhere}).$$

This completes the proof. ■

Proof of theorems 3.1 and 3.5. These follow directly from theorem 3.7. ■

Proof of proposition 3.2. Without loss of generality, suppose that player 2 is risk-neutral. Then

$$u_2(x - w(x)) \equiv x - w(x).$$

Hence by theorem 3.1, we can write

$$u_1'(\tilde{w}(x)) = \frac{\mathrm{E}(u_1(\tilde{w}(x)))}{\mathrm{E}(x - \tilde{w}(x))}.$$

Since player 1 is risk-averse, u_1' is monotonic and therefore invertible. Thus we can write

$$\tilde{w}(x) = (u_1')^{-1}(\alpha) \quad (F - \text{almost everywhere})$$

where α is the constant defined by:

$$\alpha := \frac{\mathrm{E}(u_1(\tilde{w}(x)))}{\mathrm{E}(x - \tilde{w}(x))}.$$

■

Proof of proposition 3.4. Suppose that the utility function of the risk-averse player is u_1 and the contract is the constant w_1. Now suppose that u_1 is replaced by u_2, where u_2 is more risk-averse than u_1 — that is $u_2 \equiv \psi(u_1)$ for some increasing concave function ψ. Let w_2 denote the contact corresponding to this Nash bargaining solution. It suffices to prove that $w_1 > w_2$.

Define a new utility function

$$v(t) := t\psi(u_1) + (1 - t)u_1,$$

and note that by theorem 4.2, the contract w for this situation solves

$$\frac{\mathrm{d}}{\mathrm{d}w} v(w) = \frac{v(w)}{\mathrm{E}(x) - w}.$$

Recall that by proposition 3.2, w is a constant in x. So, we can think of the optimum contract w as a function of t. It is easy to show that this function is well-defined. We can differentiate w implicitly with respect to t to get

$$\begin{aligned}\psi'(u_1(w))u_1'(w) + t\psi''(u_1(w))(u_1'(w))^2\frac{\mathrm{d}w}{\mathrm{d}t} + t\psi'(u_1(w))u_1''(w)\frac{\mathrm{d}w}{\mathrm{d}t} + (1-t)u_1''(w)\frac{\mathrm{d}w}{\mathrm{d}t} - u_1'(w)& \\ = \frac{\psi'(u_1(w))u_1'(w)\frac{\mathrm{d}w}{\mathrm{d}t}(\mathrm{E}(x) - w) + \psi(u_1(w))\frac{\mathrm{d}w}{\mathrm{d}t}}{(\mathrm{E}(x) - w)^2}&.\end{aligned}$$

Rearranging this we get

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}t}\left[t\psi''(u_1(w))(u_1'(w))^2 + t\psi'(u_1(w))u_1''(w) + (1-t)u_1''(w) - \frac{\psi'(u_1(w))u_1'(w)}{\mathrm{E}(x) - w} - \frac{\psi(u_1(w))}{(\mathrm{E}(x) - w)^2}\right]& \\ = (1 - \psi'(u_1(w)))u_1'(w)&.\end{aligned}$$

We know that the set

$$\{u_i(w(t)) : t \in [0, 1]\}$$

is a compact subset of strictly positive real numbers, since $w(t)$ must maximise a Nash product. Hence this set contains a nonzero minimum α. The number $\psi'(\alpha) \geq \psi'(u_i(w(t))) > 0$ for $t \in [0, 1]$. Hence, by using axiom (AM) and dividing ψ by $\psi'(\alpha)$, we can assume without loss of generality that $\psi'(w(t)) < 1$ for $t \in [0, 1]$. From this, we can unambiguously infer that

$$\frac{\mathrm{d}w}{\mathrm{d}t} < 0.$$

Therefore, we conclude that $w(0) < w(1)$ — that is, w_1 is greater than w_2. ■

Proof of proposition 3.6. The proof is similar to that of proposition 3.4. ■

References

Borch, K (1962), "Equilibrium in a reinsurance market," *Econometrica*, 30(3), 424–444.

Harsanyi, J C (1997), *A Bargaining Model for the Cooperative n-Person Game.*, 482 – 512, Elgar Reference Collection.

Muthoo, A (1999), *Bargaining Theory with Applications*, Cambridge University Press.

Nash, J (1950), "The bargaining problem," *Econometrica*, 18(2), 155–162.

Zeidler, E (1995), *Applied Functional Analysis: Main Principles and Their Applications*, vol 109 of *Applied Mathematical Sciences*, Springer-Verlag.